University of California
Los Angeles

Investigating Transitions in
Planetary Dynamo Models

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Geophysics and Space Physics

by

Krista Marie Soderlund

2011
The dissertation of Krista Marie Soderlund is approved.

________________________________________

Raymond J. Walker

________________________________________

Gerald Schubert

________________________________________

Paul H. Roberts

________________________________________

Jonathan M. Aurnou, Committee Chair

University of California, Los Angeles

2011
To my parents and to Chris
# Contents

1 Introduction to Planetary Magnetic Fields 1

1.1 Internal Structures 2

1.2 Magnetic Field Characteristics 5

1.3 Active Dynamos 7

1.3.1 Earth 7

1.3.2 Mercury 10

1.3.3 Ganymede 11

1.3.4 Jupiter 12

1.3.5 Saturn 13

1.3.6 Uranus and Neptune 14

1.4 Extinct Dynamos: Mars and the Moon 15

1.5 No Dynamos 18

1.6 Comparative Planetology 19

2 Modeling Convection and Dynamo Action 22

2.1 Governing Equations 22

2.1.1 Boussinesq Approximation 23

2.1.2 Boundary Conditions 27

2.2 Non-dimensional Parameters 28

2.3 Fundamentals 35
2.4 Simulating Convection and Dynamo Action .......................... 40
  2.4.1 Numerical Model ................................................. 40
  2.4.2 Numerical Method ............................................... 42
  2.4.3 Model Limitations ............................................... 43

3 Survey of Terrestrial-style Dynamo Models 47
  3.1 Introduction ......................................................... 47
  3.2 Numerical Model ................................................... 48
  3.3 Convective Heat Transfer ......................................... 54
  3.4 The Weak Influence of Magnetic Fields in Planetary Dynamo Models 57
    3.4.1 Introduction ................................................... 57
    3.4.2 Behavioral Regimes ............................................ 58
    3.4.3 Parameterization of Magnetic Field Influence ................. 76
    3.4.4 Breakdown of Dipolar Magnetic Field Generation ............. 83
    3.4.5 Conclusions .................................................... 84

4 Models of Ice Giant-style Dynamos 86
  4.1 Introduction ......................................................... 86
  4.2 Ice Giant Observations ............................................. 87
  4.3 Dynamical Regimes .................................................. 95
    4.3.1 Gas Giant-style Dynamics ..................................... 95
    4.3.2 Ice Giant-style Dynamics .................................... 98
# List of Figures

1. Internal structures of planets. .............................................. 2
2. Radial magnetic field of Earth. ........................................... 7
3. Radial magnetic fields of planets other than Earth with active dynamos. 9
4. Radial magnetic fields of Mars and the Moon. .......................... 17
5. Schematic of the dynamo model geometry. ............................. 30
6. Illustration of the $\Omega$-effect and $\alpha$-effect. ....................... 39
7. Boundary layer thicknesses versus the transition parameter. ......... 56
8. Dipolarity versus the Rayleigh number. .................................. 60
9. Radial magnetic fields and isosurfaces of axial vorticity for select models. 61
10. Axial vorticity columnarity and relative axial helicity versus the Rayleigh number. ....................................................... 65
11. Convective flow speed and heat transfer efficiency versus the Rayleigh number. ......................................................... 68
12. Integrals of rms forces versus the Rayleigh number for the dynamo models. ................................................................. 71
13. Integrals of rms forces versus the Rayleigh number for the non-magnetic models. .......................................................... 72
14. Zonal flows for select models. ............................................. 73
15. Integrals of rms azimuthal axisymmetric forces versus the Rayleigh number. ................................................................. 75
Comparison of calculated and parametrized Lorentz to Coriolis force ratios versus the Rayleigh number. ................................................................. 77
Surface observations of zonal flows, heat fluxes, and radial magnetic fields of the ice giants and Jupiter. ................................................................. 89
Internal structure schematics of Uranus and Neptune. .................. 92
Zonal flows in the Heimpel et al. (2005) model. ......................... 96
Convective characteristics in cylindrical and planar geometries. . 102
Kinetic energy components of ice giant-style models. ............... 111
Observed and simulated zonal flows of the ice giants. ............... 112
Zonal flows and meridional circulations of the ice giant-style models. 114
Comparison of zonal flow profiles to theoretical predictions. ........ 116
Heat transfer characteristics of strongly-forced thick shell models. 120
Heat transfer characteristics of strongly-forced thin shell models. 121
Snapshots of radial magnetic and velocity fields of ice giant-style models. 123
Simulated zonal flows, heat fluxes, and radial magnetic fields for the Neptune-like model. ................................................................. 125
Simulated zonal flows, heat fluxes, and radial magnetic fields for the Uranus-like model. ................................................................. 127
Comparison of observed versus simulated magnetic power spectra. 129
Internal dipolarity and quadrupolarity versus the magnetic Prandtl number. ......................................................................................... 138
Magnetic power spectra as a function of the magnetic Prandtl number. 140

Magnetic power spectra up to degree and order 10 as a function of the magnetic Prandtl number. 141

Magnetic field structures for select strongly-forced models. 143

Radial magnetic fields and isosurfaces of axial vorticity for select strongly-forced dynamo models. 144

Kinetic energy components as a function of the magnetic Prandtl number. 145

Characteristic zonal flows and meridional circulations as a function of the magnetic Prandtl number. 148

Heat flux profiles as a function of the magnetic Prandtl number. 150

Geometry of zonal torque balance. 152

Instantaneous torque balance for the $Pm = 1$ and $Pm = 2$ models. 156

Time-averaged torque balance for the $Pm = 1$ and $Pm = 2$ models. 157

Ratio of mean signed to mean unsigned magnetic field strengths and average latitude of the dipole as a function of the magnetic Prandtl number. 161

Time series of the dipole latitude for select strongly-forced dynamos. 162
List of Tables

1  Summary of physical properties and magnetic field characteristics. . . 3
2  Summary of non-dimensional parameter definitions. . . . . . . . . . 29
3  Summary of properties of the planets’ dynamo regions. . . . . . . . 32
4  Dimensionless parameter estimates for the planets’ dynamo regions. . 34
5  Summary of diagnostic parameters. . . . . . . . . . . . . . . . . . . 46
6  Input and output parameters for terrestrial survey. . . . . . . . . . 52
7  Summary of hyperdiffusion parameters used in the terrestrial survey. 53
8  Estimates of flux Rayleigh numbers for the dynamo regions of the giant
    planets. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 104
9  Summary of ice giant-style dynamo case study simulations. . . . . . 107
10 Summary of strongly-forced dynamo $P_m$-study simulations. . . . . 136
ACKNOWLEDGEMENTS

First, I would like to thank my advisor, Jonathan Aurnou, who has guided and supported me through this journey. Jon has patiently allowed me to work through problems and has always been available for ideas and advice. I am continually inspired by his passion and enthusiasm.

I thank my thesis committee – Paul Roberts, Jerry Schubert, Ray Walker – for devoting their time and for offering their suggestions on how to improve this research. Jerry and I also collaborated on a paper reviewing planetary magnetism (Schubert and Soderlund, 2011), and I am thankful to have had this opportunity. Portions of the manuscript have been adapted for inclusion in this dissertation, including the introduction to planetary magnetic fields, the description of governing equations and non-dimensional parameters, the discussion of giant planet models in the literature, and the comments on the future outlook of the field.

Johannes Wicht, who kindly supplied the numerical code I used and helped resolve any issues that arose, and Jerome Noir, who taught me how to use the code, are also owed my sincerest thanks. Moritz Heimpel also offered useful advice on running the code and analyzing its output. The staff at the San Diego Supercomputing Center and the NASA Advanced Supercomputing Division have helped me port the code and maximize its efficiency.

My thanks also go to collaborator and former officemate Eric King, with whom I
have co-authored two papers, King et al. (2010) and Soderlund et al. (2011), that are included in this dissertation. Our first paper, which is only briefly summarized in this document, investigates convective heat transfer in terrestrial-style planetary dynamo models. I planned, carried out, and analyzed all of the ‘dynamo subset’ simulations which allowed us to test our hypothesis that boundary layers control heat transfer. This project also showed that magnetic fields have only a second order influence on convective heat transfer; towards explaining this result, we wrote a follow-up paper, in review at Earth Planet. Sci. Lett. I designed, executed, and analyzed a suite of simulations to quantify and explain the influence of magnetic fields on convective dynamics and wrote the paper. I am thankful to Eric for his ready exchange of ideas, his excellent writing and editing skills, and his patience through these projects.

My officemates in 5652 and many fellow graduate students have created a stimulating and enjoyable work environment. Mike Calkins, Jonathan Cheng, Mike Hartinger, Jared Leisner, Colleen Milbury, Carrie Nugent, and Britney Schmidt deserve special recognition as they have enhanced my time at UCLA both personally and professionally over the years.

This thesis is dedicated to my family – my parents, Robin and Wayne, my brother, Luke, and my significant other, Chris – who have offered their unending faith, support, and love. As this chapter of life closes, I am sure that we all are ready to for me to no longer worry about ‘failing out of school’.

Funding was provided by the National Defense Science and Engineering Graduate
(NDSEG) Fellowship, the NASA Planetary Atmospheres Program through Grants NNG06GD12G, NNX09AB61G, and NNX09AB57G, and the National Science Foundation through Grant AAG0909206. This support is gratefully acknowledged.
Vita

October 28, 1982  Born, Duluth, Minnesota

2005  B.S., Physics
       Florida Institute of Technology
       Summa Cum Laude

2005  B.S., Space Sciences
       Florida Institute of Technology
       Summa Cum Laude

2006–2011  Graduate Student Researcher,
            University of California, Los Angeles

2009  M.S., Geophysics and Space Physics
       University of California, Los Angeles

2009–2010  Teaching Assistant,
           University of California, Los Angeles

2011–2013  Postdoctoral Fellow
           University of Texas, Austin

Publications


Abstract of the Dissertation

Investigating Transitions in Planetary Dynamo Models

by

Krista Marie Soderlund

Doctor of Philosophy in Geophysics and Space Physics

University of California, Los Angeles, 2011

Professor Jonathan M. Aurnou, Chair

All planets in the solar system have or once had intrinsic magnetic fields, with the possible exception of Venus. The properties and characteristics of these fields are as diverse as the planets themselves. Given this diversity, the fundamental goal is to determine what controls the strength, morphology, and evolution of planetary magnetic fields. Since these fields are thought to result from dynamo action driven by thermochemical convection in electrically-conducting fluid regions, the coupling between magnetic fields, fluid flow, and heat/mass transfer must also be understood. We seek to investigate this coupling and to understand better the processes that occur in numerical dynamo models and, hopefully, in planetary cores as well.

The magnetic fields of planets and stars are often thought to play an important role in
the fluid motions responsible for field generation. It is typically argued that magnetic
fields will fundamentally alter the convective dynamics when $\Lambda_i \gtrsim 1$, where the
traditional Elsasser number, $\Lambda_i$, characterizes the relative strengths of the Lorentz and
Coriolis forces. This change, however, does not occur in many magnetoconvection and
dynamo studies, despite having strong magnetic fields (e.g., Olson and Glatzmaier,
1995; Zhang, 1995; Kageyama and Sato, 1997; Christensen et al., 1999; Zhang and
Schubert, 2000; Jones, 2007; Jault, 2008; King et al., 2010). These results imply that
the traditional force balance argument from linear analysis using the Elsasser number
$\Lambda_i$ may not be an adequate measure of the dynamical influence of the Lorentz force
in convection systems.

I have carried out a suite of dynamo and non-magnetic, but otherwise identical,
models which are compared in order to quantify the influence of magnetic fields on
convective dynamics systematically and to understand why the Lorentz force has a
surprisingly weak dynamical role in magnetic systems. The characteristics of con-
vection, including convective flow structures and speeds as well as heat transfer, are
found to be only weakly affected by the presence of magnetic fields. We compare
different parameterizations of the relative influence of magnetic and rotational forces
and show that the traditional Elsasser number overestimates the role of the Lorentz
force in dynamos. Instead, we argue that an alternatively defined ‘dynamic Elsasser
number’ better represents the Lorentz to Coriolis force ratio. We also find a sharp
transition between dipolar and multipolar dynamos. This morphological transition is
linked to the breakdown of helical flow as inertial forces become stronger than viscous forces. Because viscous forces are negligible in planetary interiors, my findings imply that present day dynamo models with moderate rotation rates ($E \gtrsim 10^{-4}$) may be too viscous to reproduce the physical mechanisms of field generation in planetary interiors correctly where viscosity is negligible.

The dichotomy between dipole-dominated gas giants and multipolar ice giants is also investigated. In addition to having dramatically different magnetic fields, these bodies also have contrasting zonal flows and thermal emissions. We show that strongly-forced Boussinesq dynamo models generate zonal flow, thermal emission, and magnetic field patterns that qualitatively agree with observations of the ice giants. Based on these modeling results, we hypothesize that inertial convection may be relevant to Uranus and Neptune. We also note that it is critical to understand what controls the transition to the inertial regime and how this transition scales to planetary settings, as well as argue that anelastic models with radially-varying density and electrical conductivity should be carried out.

Understanding the influence of the magnetic Prandtl number is important for applying the results of numerical models to planetary bodies since computational limitations restrict models to $Pm$ values much larger than those expected for planetary interiors. Towards this end, we carry out a suite of dynamo simulations to measure the effects of changing the magnetic Prandtl number on the magnetic and velocity fields in the inertially-dominated convective regime. We find that the fluid dynamical, heat...
transfer, and dynamo characteristics all fundamentally change with $Pm$. This result implies that we have not reached the asymptotic limit in terms of the magnetic Prandtl number and caution must be applied when extrapolating model results to planetary settings.
1 Introduction to Planetary Magnetic Fields

With the possible exception of Venus, all planets in the solar system have or once had internally generated magnetic fields. The Moon also had its own magnetic field in the past, and Jupiter’s satellite Ganymede has a magnetic field at present. The angrite meteorite parent body has also generated a magnetic field in its history (Weiss et al., 2008). Further, it is expected that many of the extrasolar planets now being detected have magnetic fields and some observations support this conjecture (e.g., Sanchez-Lavega, 2004; Shkolnik et al., 2005, 2008).

While many planets, satellites, and small bodies generate magnetic fields by dynamo action in their interiors, not all objects with the requisite electrically conducting fluid layers have a dynamo. Prominent among such bodies in our own solar system are Venus and Io, which are likely to have completely or at least partially molten metallic cores (e.g., Konopliv and Yoder, 1996; McEwen et al., 1989; Khurana et al., 2011). The cores in Mars and the Moon are also likely partially molten at present (e.g., Yoder et al., 2003; Khan et al., 2004), yet the dynamos that once existed in these bodies have ceased to function.

A planetary magnetic field, or the absence of such a field, informs us about the internal structure of a body and its thermal evolution. Together with the gravitational field, the magnetic field provides a window into the interiors of bodies that can only be probed from a distance. The magnetic fields recorded and preserved in the crusts
of planets and satellites also provide a window into the histories of the bodies. Planetary magnetism therefore merits study from many perspectives, including that of understanding the physics of how dynamos work and what they imply about the past and present properties of the body containing the dynamo.

1.1 Internal Structures

The planets in our solar system can be classified into three types – terrestrial, gas giant, or ice giant – based on their physical properties, and the magnetic fields show a large diversity in structure and strength. Figure 1 shows representative internal structure schematics for each type of planet, and Table 1 summarizes the physical properties and magnetic field characteristics of the planets and select satellites.

![Internal Structure Schematics](image)

**Figure 1:** Internal structure schematics. Differentiated icy satellites have terrestrial-like internal structures below their outer ice shell. Estimates of the layer radii for select bodies are given in Table 3. Color gradients represent density stratification. Not to scale.
<table>
<thead>
<tr>
<th>Planet</th>
<th>Mass(^{1}) (10(^{24}) kg)</th>
<th>Radius(^{1}) (km)</th>
<th>Density(^{1}) (kg/m(^3))</th>
<th>MOI(^{-2-4})</th>
<th>Dipole moment ((\mathcal{M}_\oplus))</th>
<th>Dipolarity</th>
<th>Dipole tilt (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>0.33</td>
<td>2440</td>
<td>5.4</td>
<td>0.33</td>
<td>0.30</td>
<td>4 × 10(^{-4})</td>
<td>&lt; 0.71</td>
</tr>
<tr>
<td>Venus</td>
<td>4.87</td>
<td>6052</td>
<td>5.2</td>
<td>0.33</td>
<td>(\gtrsim 10^{-3})</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Earth</td>
<td>5.97</td>
<td>6371</td>
<td>5.5</td>
<td>0.33</td>
<td>38</td>
<td>1</td>
<td>0.61</td>
</tr>
<tr>
<td>Moon</td>
<td>0.07</td>
<td>1738</td>
<td>3.3</td>
<td>0.39</td>
<td>(\gtrsim 100)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Mars</td>
<td>0.64</td>
<td>3390</td>
<td>3.9</td>
<td>0.37</td>
<td>(\gtrsim 0.1)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Jupiter</td>
<td>1900</td>
<td>69,911</td>
<td>1.3</td>
<td>0.25</td>
<td>550</td>
<td>2 × 10(^{4})</td>
<td>0.61</td>
</tr>
<tr>
<td>Io</td>
<td>0.09</td>
<td>1821</td>
<td>3.5</td>
<td>0.38</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Europa</td>
<td>0.05</td>
<td>1565</td>
<td>3.0</td>
<td>0.35</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Ganymede</td>
<td>0.15</td>
<td>2634</td>
<td>1.9</td>
<td>0.31</td>
<td>0.91</td>
<td>0.95</td>
<td>4</td>
</tr>
<tr>
<td>Callisto</td>
<td>0.11</td>
<td>2403</td>
<td>1.9</td>
<td>0.35</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Saturn</td>
<td>570</td>
<td>58,232</td>
<td>0.7</td>
<td>0.21</td>
<td>28</td>
<td>600</td>
<td>0.85</td>
</tr>
<tr>
<td>Titan</td>
<td>0.13</td>
<td>2575</td>
<td>1.9</td>
<td>0.34</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Uranus</td>
<td>87</td>
<td>25,362</td>
<td>1.3</td>
<td>0.23</td>
<td>32</td>
<td>50</td>
<td>0.42</td>
</tr>
<tr>
<td>Neptune</td>
<td>100</td>
<td>24,624</td>
<td>1.6</td>
<td>0.23</td>
<td>27</td>
<td>25</td>
<td>0.31</td>
</tr>
</tbody>
</table>

Table 1: Summary of physical properties and magnetic field characteristics. Overbars indicate mean values. Physical property data taken from \(^{1}\)Lodders and Fegley (1998), \(^{2}\)de Pater and Lissauer (2001), \(^{3}\)Schubert et al. (2007) for the MOI of the Galilean satellites, \(^{4}\)Iess et al. (2010) for the MOI of Titan, and \(^{5}\)Kivelson and Bagenal (1999) for the magnetic dipole moment measured with respect to the Earth’s value, \(\mathcal{M}_\oplus = 7.9 \times 10^{15} \ T \ m^3\). Other magnetic field characteristics are calculated using Equations (2) – (4) and data in Uno et al. (2009) for Mercury, IAGA Division V-MOD Geomagnetic Field Modeling website for Earth, Yu et al. (2010) for Jupiter, Kivelson et al. (2002) for Ganymede, Burton et al. (2009) for Saturn, and Holme and Bloxham (1996) for the ice giants.
The mean density and moment of inertia (MOI) of a planet provide constraints on the internal composition and structure. All of the planets and many of the satellites show evidence of differentiation into layers. The terrestrial planets have an outer rocky crust, an intermediate silicate mantle, and a central iron-alloy core (e.g., Riner et al., 2008; Zharkov, 1983; Dziewonski and Anderson, 1981; Longhi et al., 1992). The gas giants, Jupiter and Saturn, have outer molecular envelopes of hydrogen and helium, fluid mantles that are largely composed of metallic hydrogen and helium, and perhaps central rocky cores (Guillot, 1999b). The ice giants, Uranus and Neptune, have outer molecular envelopes of hydrogen, helium, and ices; intermediate ionic oceans of water, methane, and ammonia; and possibly central rocky cores (Guillot, 1999b). However, the layer thicknesses are not well-constrained for planets other than the Earth. Outer solar system satellites exhibit a wide variety of sizes, from tens of kilometers to larger than the planet Mercury, and have mean densities that are icy to rocky. Ganymede is the largest satellite in the solar system and also has the lowest moment of inertia of all known planetary bodies (Schubert et al., 2007). This indicates that the satellite is fully differentiated, with a thick water-ice shell overlying an intermediate rocky mantle and a central metallic core. Thus, Ganymede has a terrestrial-style internal structure beneath the ice shell.
1.2 Magnetic Field Characteristics

Magnetic fields originate in the electrically conducting fluid regions of planets and satellites and are likely driven by dynamo action (e.g., Elsasser, 1939; Bullard, 1949; Stevenson, 2003). These regions are expected to be unstable to thermochemical convection in order to remove the heat of formation and radioactive decay (Stevenson, 2010). Convection of an electrically conducting fluid drives electrical currents and generates a magnetic field; the process whereby kinetic energy is converted into magnetic energy is known as dynamo action. While convection is the most probable dynamo driver, it has been argued that precession, libration, nutation, and tidal interactions may also promote dynamo action (Malkus, 1994; Tilgner, 2007; Noir et al., 2009; Herreman et al., 2009; Dwyer et al., 2010).

While magnetic fields are driven in the planets' fluid interiors, observations can only be made at the surface and from orbit around the planet. The measured potential field has diffused outward through the overlying electrically insulating or less electrically conducting regions. This field can be expressed as $\mathbf{B} = -\nabla \Phi$, where the magnetic potential $\Phi$ can be decomposed into spherical harmonics:

$$\Phi(r, \theta, \phi) = R_P \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left\{ \left( \frac{R_P}{r} \right)^{l+1} \left[ g_l^m \cos(m\phi) + h_l^m \sin(m\phi) \right] P_l^m(\cos \theta) \right\}$$

(1)

where $r$ is radius, $\theta$ is colatitude, $\phi$ is longitude, $R_P$ is the planet radius, $l$ is spherical harmonic degree, $m$ is spherical harmonic order, $P_l^m$ are associated Legendre polynomials with partial Schmidt normalization (see Appendix B of Merrill et al. (1996)).
and $g_l^m$ and $h_l^m$ are the gauss coefficients. The gauss coefficients are used to describe planetary magnetic fields. The radial magnetic field for a given radius $r$ is obtained through

$$ B_r(r) = -\nabla \Phi \cdot \hat{r} = \sum_{l=1}^{l_{\text{max}}} \sum_{m=0}^{l} \left\{ (l+1) \left( \frac{R_P}{r} \right)^{l+2} \left[ g_l^m \cos(m\phi) + h_l^m \sin(m\phi) \right] P_l^m (\cos \theta) \right\} $$

(2)

where $l_{\text{max}}$ is the maximum harmonic degree of the observations. The strength of the dipole relative to the total field strength is the dipolarity,

$$ f = \frac{\sum_{m=0}^{1} \left[ |g_l^m| + |h_l^m| \right]}{\sum_{l=1}^{l_{\text{max}}} \sum_{m=0}^{l} \left[ |g_l^m| + |h_l^m| \right]}. $$

(3)

Perfectly-dipolar magnetic fields have $f = 1$ by convention. The fields are considered to be dipole-dominated when $f > 0.5$. The tilt of the dipole relative to the rotation axis is given by

$$ \theta_d = \tan^{-1} \left( \frac{\left[ (g_1^1)^2 + (h_1^1)^2 \right]^{1/2}}{|g_1^0|} \right). $$

(4)
1.3 Active Dynamos

In this section, the magnetic fields of planets with active dynamos are reviewed.

1.3.1 Earth

Earth’s magnetic field is at least 3.5 billion years old as recorded by crustal rocks that were magnetized as they cooled (Hale and Dunlop, 1984; Tarduno et al., 2007). These paleomagnetic records indicate that the dynamo has almost always had a strong axial dipole component with a field strength that has not significantly varied over time (Tarduno et al., 2007; Hulot et al., 2010). The present field, shown in Figure 2, has a mean surface strength of about 40 $\mu$T and the dominant dipole component is offset $10^\circ$ from the rotation axis. However, episodic reversals in field polarity have been

![Figure 2](image_url)

Figure 2: Radial magnetic field of Earth at the a) surface and b) core-mantle boundary (CMB). Colors represent field intensity where purple (green) indicates outward (inward) directed field. A mollweide projection is used where horizontal lines indicate constant latitude. IGRF-11 coefficients are used and available on the IAGA Division V-MOD Geomagnetic Field Modeling website.
recorded as magnetic stripes in oceanic crust over the past hundred million years (Johnson et al., 2003). During the ∼10,000 years over which reversals occur, the total field strength decreases and the field becomes multipolar, then establishes a strong reversed dipolar field (Bogue and Merrill, 1992).

Observatories and satellites have greatly improved our understanding of the Earth’s magnetic field and its evolution over short timescales. The Magsat (1979–1980), Ørsted (1999–), CHAMP (2000–2010), and SAC-C (2000–2004) satellites have provided continuous measurements of the field over a range of altitudes for more than ten years. The SWARM satellite constellation will be launched in 2012 to continue these efforts.

Figure 2 shows the radial magnetic field at the surface and the core-mantle boundary plotted up to spherical harmonic degree 13. Higher order harmonics of the internal field are masked by crustal remnant magnetism. The dynamic nature of the geodynamo is revealed through secular variation, where changes in the strength and spatial structure of the field occur over timescales of months (Olsen and Mandea, 2008) to centuries (Bloxham et al., 1989) to millenia (Cande and Kent, 1995). The most notable aspects of secular variation are polarity reversals, westward drift of the non-axisymmetric field, anticyclonic field motions at high latitudes, growth of the South Atlantic anomaly where the field is unusually weak, and geomagnetic jerks (Finlay et al., 2010). The physical mechanisms that lead to these behaviors, however, are not yet understood.
Figure 3: Radial magnetic field at the surfaces of planets other than Earth with active dynamos.

Data taken from Uno et al. (2009) for Mercury (with spectral resolution $l, m \leq 3$), Kivelson et al. (2002) for Ganymede ($l, m \leq 2$), Yu et al. (2010) for Jupiter ($l, m \leq 3$), Burton et al. (2009) for Saturn ($l, m \leq 3$), and Holme and Bloxham (1996) for the ice giants ($l, m \leq 3$).
1.3.2 Mercury

Mercury has the weakest intrinsic magnetic field in the solar system with a mean surface strength of only about 0.3 $\mu T$. This field was first measured during the Mariner 10 flybys in 1974 and 1975 (Ness et al., 1975, 1976) and again during the MESSENGER flybys in 2008 and 2009 (Anderson et al., 2008, 2010; Uno et al., 2009; McNutt et al., 2010). These observations suggest that the field is dipole-dominated and nearly aligned with the rotation axis. The surface radial magnetic field plotted up to spherical harmonic degree three is shown in Figure 3a. However, recent results from MESSENGER show that the planet also has a significant axial quadrupole component since the dipole is located about 0.16 Mercury radii above the planet’s center (Anderson et al., 2011). Note that data from Uno et al. (2009) is used in Figure 3 and Table 1 since only the $g_{10}$ and $g_{20}$ gauss coefficients are given by Anderson et al. (2011). Compared to the Earth, the field is about 100 times weaker at the surface and about 500 times weaker at the core-mantle boundary, assuming Mercury’s core has a radius of 1900 km.

Mercury’s magnetic field is thought to be generated by dynamo action because libration measurements (Margot et al., 2007), MESSENGER gravity measurements (Smith et al., 2010), and thermodynamic phase equilibrium simulations of Malavergne et al. (2010) imply the presence of a liquid core. It is not yet understood, however, why Mercury has such a weak magnetic field or why the field has a strong quadrupole component. Further, MESSENGER found that the internal field interacts with currents
in the planet’s magnetosphere (McNutt et al., 2010), but the dynamical implications of this coupling are not well constrained (Glassmeier et al., 2007; Kabin et al., 2008; Gomez-Perez and Solomon, 2010; Gomez-Perez and Wicht, 2010).

1.3.3 Ganymede

The Galileo spacecraft revealed that the Jovian satellite Ganymede also has an intrinsic magnetic field (Kivelson et al., 1996). As shown in Figure 3b, these measurements suggest that the field is nearly axially-aligned and dipole-dominated with a mean surface strength of about 1 µT. However, it might also be appropriate to consider the value of Ganymede’s magnetic field at the ice-rock boundary since the body has a terrestrial-style internal structure beneath the ice shell. Assuming an ice shell thickness of about 900 km (Schubert et al., 2007), the mean magnetic field is 3 µT at the ice-rock boundary. This is about an order of magnitude larger than Mercury’s surface field and an order of magnitude smaller than Earth’s surface field. The field resolution, however, is limited to the dipole and quadrupole components and additional data are needed to characterize the magnetic field further.

Since Ganymede is thought to have a metallic core with a fluid component, the field is likely driven by dynamo action (Schubert et al., 1996; Sarson et al., 1997; Kivelson et al., 2002). The size of the core is poorly constrained, and the energy source for maintaining a liquid or partially liquid core in such a small body is not well understood (Hauck et al., 2006; Bland et al., 2008).
1.3.4 Jupiter

The magnetic field of Jupiter was discovered when Burke and Franklin (1955) showed that the planet was a strong radio source. This was the first detection of a planetary magnetic field besides that of the Earth. The existence of the field was confirmed when the planet was visited by the Pioneer 10 flyby in 1973 (Smith et al., 1974). Additional measurements were obtained by the Pioneer 11 flyby in 1974 (Smith et al., 1975), the Voyager 1 and 2 flybys in 1979 (Ness et al., 1979), the Ulysses flybys in 1992 and 2004 (Balogh et al., 1992), the Galileo orbiter between 1995 and 2003 (e.g., Connerney, 1993; Yu et al., 2010), and the New Horizons flyby in 2007 (e.g., McComas et al., 2007; McNutt et al., 2007).

These measurements show that the magnetic field is dominated by the axial dipole component and is the strongest field in the solar system with a mean surface strength of 550 $\mu$T and dipole moment 20,000 times that of the Earth. The surface of gaseous planets is taken to be the 1 bar pressure level. The surface radial magnetic field including up to the octupole component is shown in Figure 3c. The spatial and temporal resolution of the field is a significant limitation. While Jupiter’s magnetic field has been measured by several missions, the orbits do not extend inward of 1.6 Jupiter radii and have longitudinal data gaps (Russell and Dougherty, 2010). Consequently, the observations are only resolved to spherical harmonic degree three. Secular variations in Jovian magnetic field strength or dipole tilt cannot be unambiguously determined over the 20 years between the Pioneer and Galileo missions (Yu et al., 2010).
The magnetic field is produced by convectively-driven dynamo action in the highly electrically conducting metallic hydrogen mantle and possibly in the less electrically conducting region near the base of the molecular envelope as well (Stanley and Glatzmaier, 2010; Heimpel and Gomez-Perez, 2011). The exact location and extent of the dynamo region is poorly constrained. Nellis (2000) argues that the dynamo can extend up to 95% of the planet radius. Consequently, the dynamo region is expected to exhibit significant changes in density and electrical conductivity, the effects of which are relatively unknown. The upcoming Juno mission will critically improve our understanding of Jupiter by resolving the magnetic field up to spherical harmonic degree 14, detecting short timescale secular variation, and constraining the planet’s internal structure (J.E.P. Connerney, private communication; Kaspi et al., 2009).

1.3.5 Saturn

Saturn was revealed to have a magnetic field during the Pioneer 11 flyby in 1979 (Smith et al., 1980), and subsequent observations were made by Voyager 1 in 1980 (Ness et al., 1981) and Voyager 2 in 1981 (Ness et al., 1982). The Cassini spacecraft arrived at Saturn in 2004 and is still collecting data (Dougherty et al., 2005; Burton et al., 2009). These observations have mapped the large-scale radial magnetic field, shown in Figure 3d, and suggest that no resolved secular variation has occurred (Burton et al., 2009). The spatial resolution is limited to spherical harmonic degree three and the axisymmetric mode due to uncertainty in the planet’s rotation rate.
(Burton et al., 2009). The field is strongly dipole-dominated with a mean surface (1 bar) strength of about 30 $\mu T$ and has a remarkably small offset of less than 1°. No other observed magnetic field has such a small dipole tilt.

According to Cowling’s theorem, no axisymmetric magnetic field can be generated by dynamo action, so the field must have unresolved non-axisymmetric components. It has also been proposed that a secondary mechanism operating in the overlying regions may be responsible for axisymmetrizing the field (e.g., Stevenson, 1980, 1982, 1983; Schubert et al., 2004; Christensen and Wicht, 2008; Stanley, 2010). The mechanisms that lead to a strongly axisymmetric magnetic field, however, remain poorly understood.

### 1.3.6 Uranus and Neptune

The magnetic fields of Uranus and Neptune were discovered by the Voyager 2 flybys in 1986 and 1989, respectively (Ness et al., 1986, 1989; Holme and Bloxham, 1996). The spacecraft passed within 4.2 Uranian radii and 1.2 Neptunian radii, allowing their dipole, quadrupole, and octupole components to be mapped as shown in Figure 3e-f.

The ice giants’ magnetic fields are fundamentally different from those of the gas giants since they are multipolar, with $f = 0.42$ for Uranus and $f = 0.31$ for Neptune. The dipole component is tilted about 59° and 47° from the rotation axis for Uranus and Neptune, respectively. Both planets have mean surface field strengths near 30 $\mu T$, comparable to that of the Earth and Saturn.
These multipolar magnetic fields were a surprise to many scientists, and it is still not understood why these bodies produce remarkably different fields from the other planets. The fields are thought to be generated in the planets’ ionic oceans (Cavazzoni et al., 1999; Lee et al., 2006). However, the internal structures and dynamics of the ice giants are poorly constrained. Models of the ice giant-style dynamos will comprise the second half of this dissertation.

1.4 Extinct Dynamos: Mars and the Moon

Mars Global Surveyor (MGS) showed that Mars does not presently have an active magnetic field, yet it discovered magnetic anomalies associated with the magnetization of crustal rocks due to an ancient dynamo (e.g., Ness et al., 1999; Acuña et al., 1998, 1999; Connerney et al., 1999, 2001; Purucker et al., 2000; Langlais et al., 2004; Mitchell et al., 2007; Lillis et al., 2004, 2008). Similarly, the Moon does not presently have an active magnetic field, but likely had one in the past. Crustal magnetization of lunar rocks was first detected during the era of Apollo exploration (Coleman et al., 1972), and has since been mapped in detail by Lunar Prospector in 1998 and 1999 (Binder, 1998; Purucker, 2008).

The crustal magnetic fields of Mars and the Moon are mapped in Figure 4. The Martian crustal field is essentially confined to the southern hemisphere of the planet and is further concentrated into a relatively small region of the southern highlands.
between about 120° and 240° East longitude (Terra Cimmeria and Terra Sirenum). Notably, many major impact basins lack magnetic signatures. Intensities of local flux patches at spacecraft altitudes tend to be one to two orders of magnitude larger than the Earth’s crustal magnetization of about 20 nT (Langlais et al., 2010). The lunar crustal field is distributed unevenly across the surface with flux patches that tend to be at least an order of magnitude weaker than those of Mars (Langlais et al., 2010).

The internal structures of these bodies are consistent with the inferred dynamo mechanism since there is strong evidence that both objects have metallic cores. The exact radii and densities of the cores are not well-known, but are constrained by the measured values of the moment of inertia of both objects (e.g., Folkner et al., 1997; Konopliv et al., 1998). The physical states of the cores are also largely unknown, although there is evidence that both cores are at least partially molten (Williams et al., 2001; Yoder et al., 2003; Khan et al., 2004).

Why then did the dynamos in Mars and the Moon turn off? For both bodies, one of two scenarios seems most likely. Either the cores of these bodies cooled to the point where thermal convection and dynamo action were no longer sustainable or they cooled to the point of growing such large solid inner cores that dynamo action could no longer take place in the surviving liquid outer core.

Modeling efforts of the Martian and lunar dynamos are reviewed in Schubert and Soderlund (2011).
Figure 4: Radial magnetic fields of a) Mars and b) the Moon. The colors represent field intensity where red (blue) indicates outward (inward) directed field. A Mercator projection is used, and circles indicate impact basins. Adapted from Langlais et al. (2010).
1.5 No Dynamos

Venus is the only planet that does not appear to presently have an active intrinsic magnetic field (Bridge et al., 1967; Russell et al., 1980; Phillips and Russell, 1987). Phillips and Russell (1987) gives an upper magnetic field strength limit of 1 nT. Why does dynamo action not occur Venus? The planet likely has a molten metallic core similar in size to Earth’s core (Zhang and Zhang, 1995). Two plausible explanations have been discussed. One possibility is that Venus has not cooled sufficiently for an inner core to have yet formed but enough that a purely thermally-driven dynamo cannot operate. This is consistent with Venus’ slightly smaller size compared with Earth (Stevenson et al., 1983). It also attributes a major role to compositional buoyancy release upon inner core solidification in driving dynamo action. A second explanation is that Venus has no plate tectonics and, therefore, is inefficient in cooling its core (Nimmo and Stevenson, 2000). A small heat flow out of Venus’ core could mean that the core is not convective, a state without the possibility of dynamo action. Venus may have had a dynamo in the past, perhaps if a sudden mantle overturn resurfaced the planet and rapidly cooled the core as has been proposed for the Moon (Stegman et al., 2003). Venus’ high surface temperature makes it unlikely (though not impossible) for crustal magnetization to have preserved the signature of a past dynamo. Absence of plate tectonics on Mercury and Ganymede, however, has not inhibited these bodies from generating magnetic fields in their cores.
Io, a satellite of Jupiter about the size of Earth’s Moon, also does not have a magnetic field (Jia et al., 2010). Yet it has a large metallic core (Schubert et al., 2007). This is perhaps surprising in that its close neighbor, Ganymede, has a magnetic field and a virtually identical internal structure below its thick ice shell. What accounts for the lack of dynamo activity in Io’s core? The answer appears to be that the satellite is too hot. There is so much heat produced in the mantle by tidal dissipation that it cannot cool its core. With no core cooling, there is no core thermal convection and no dynamo activity (Wienbruch and Spohn, 1995; Khurana et al., 2011). This is basically the same explanation as offered above for why Venus has no magnetic field, though the reason for inefficient cooling of the core is different in the two cases.

Callisto and Titan, satellites of Jupiter and Saturn, respectively, might be expected to have magnetic fields based on their comparable sizes to Ganymede. However, they do not (Jia et al., 2010). The lack of a magnetic field is easy to understand in the case of these bodies because they are only partially differentiated, rock from ice, and do not have metallic cores (Schubert et al., 2004; Iess et al., 2010).

1.6 Comparative Planetology

While there are outstanding questions about each planetary magnetic field, it is also valuable to consider the comparative approach which attempts to explain the similarities and differences among the fields. Given the diverse spectrum, the fundamental
goal is to determine what controls the strength, morphology, and evolution of the magnetic fields. Since dynamos are intimately linked to convective fluid motions driven by thermal and compositional gradients, the coupling between magnetic fields, fluid flow, and heat/mass transfer must also be understood. I have carried out surveys of terrestrial-style and ice giant-style numerical dynamo models in order to investigate this coupling.

The terrestrial-style survey shows that magnetic fields tend to have a second order influence on convective dynamics and heat transfer. This finding applies only to the asymmetric convective motions and not to the zonal flows. We measure the competition between Lorentz, Coriolis, inertial, and viscous forces and show that the Lorentz force is never dominant. This subdominance explains why present day dynamo and non-magnetic models exhibit similar convective (non-zonal) behaviors. We further demonstrate that a dynamic Elsasser number accurately parametrizes the Lorentz to Coriolis force ratio. Moreover, our results also suggest that the transition between dipolar and multipolar magnetic fields is due to the breakdown of helical flow, which occurs when inertial forces become stronger than viscous forces and is, therefore, a hydrodynamic process.

The ice giant-style survey shows that dynamos characterized by poorly organized, three-dimensional convective motions can generate zonal flows, thermal emission patterns, and magnetic fields that are similar to those observed on Uranus and Neptune. This result suggests that inertially-dominated convection should be further investi-
gated towards explaining the dynamics of the ice giants.

Furthermore, we show that increasing the electrical conductivity of the fluid in strongly-forced dynamo models generates strong, small-scale magnetic fields that can cause the zonal flow direction to reverse compared to a non-magnetic model with otherwise identical hydrodynamic control parameters. In contrast, decreasing the fluid’s electrical conductivity generates weak magnetic fields that are dominated by the axial quadrupole component.
2 Modeling Convection and Dynamo Action

2.1 Governing Equations

The governing equations of dynamo theory are the induction equation, conservation of mass, conservation of momentum, conservation of energy, and the equation of state. The unknowns are the magnetic field $B$, the velocity field $u$, and three thermodynamic variables, e.g., pressure $p$, temperature $T$, and density $\rho$. A detailed discussion and derivation of these equations is given in Davidson (2001).

The induction equation is

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) - \nabla \times (\eta \nabla \times B)$$  \hspace{1cm} (5)

where $\eta = 1/\mu_0 \sigma$ is the magnetic diffusivity, $\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of free space, and $\sigma$ is the electrical conductivity. The conservation of mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$  \hspace{1cm} (6)

and the conservation of momentum is

$$\rho \frac{Du}{Dt} + 2\rho \Omega \times u = -\nabla p - \rho \Omega \times (\Omega \times r) + J \times B + \rho g + \frac{1}{3} \rho \nu \nabla (\nabla \cdot u) + \rho \nu \nabla^2 u,$$  \hspace{1cm} (7)

where $\Omega$ is the rotation rate of the reference frame, $r$ is the position vector, $g$ is the acceleration of gravity, and $\nu$ is the kinematic viscosity. In (7), terms from left to right are inertial acceleration, Coriolis acceleration, pressure gradient, centrifugal force, Lorentz force, buoyancy, and two viscous diffusion terms where the Stokes
assumption to neglect the bulk viscosity has been made (e.g., Kundu and Cohen, 2002). The conservation of energy equation is

$$\frac{DT}{Dt} - \frac{\alpha T D\rho}{\rho C_p Dt} = \frac{\Phi + \nabla \cdot (k \nabla T)}{\rho C_p},$$

(8)

where $\alpha$ is the thermal expansivity, $C_p$ is the specific heat at constant pressure, $k$ is the thermal conductivity, and $\Phi$ is the dissipation function including both viscous and ohmic heating. The second term on the left of (8) represents heating (cooling) due to adiabatic compression (expansion).

The final equation of state relates density to the temperature and/or pressure: $\rho = \rho(T, P)$. This relationship is determined experimentally, from first principle calculations, and by molecular dynamic simulations (e.g., Anderson and Ahrens, 1994; Laio et al., 2000; Nettelmann et al., 2008). However, it is often simplified in the Boussinesq and anelastic approximations discussed below.

### 2.1.1 Boussinesq Approximation

In most geodynamo studies, (5)–(8) are simplified using the Boussinesq approximation. Here, the material properties are assumed to be constant and the density is treated as constant except in the buoyancy force of the momentum equation. The induction equation simplifies to

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \eta_o \nabla^2 B,$$

(9)
the conservation of mass simplifies to

\[ \nabla \cdot \mathbf{u} = 0, \]  

(10)

and the conservation of momentum simplifies to

\[ \frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} = -\nabla \Pi + \frac{1}{\rho_o} \mathbf{J} \times \mathbf{B} - \alpha_o T' \mathbf{g} + \nu_o \nabla^2 \mathbf{u}. \]  

(11)

The buoyancy force in (11) arises from density differences with respect to the density of the motionless background state (assumed to be constant density \( \rho_o \)). These perturbations, \( \rho' \), are produced by temperature variations, \( T' \), from the background state temperature profile. The equation of state is then given by \( \rho = \rho_o + \rho' = \rho_o(1 - \alpha_o T') \), and the gravity term becomes \( \rho \mathbf{g} = \rho_o \mathbf{g} - \rho_o \alpha_o T' \mathbf{g} \). The mean gravitational component is hydrostatic and can be absorbed into the pressure gradient term. Since the centrifugal force can be written as the gradient of a potential, this term is also incorporated into the pressure gradient. As a result, \( \Pi \) represents the effective pressure relative to the hydrostatic pressure of a motionless background state, i.e., \( \Pi \) is associated with the velocity field. The energy equation also simplifies in the Boussinesq approximation. Heating associated with adiabatic compression and due to viscous and ohmic dissipation are assumed negligible (Hewitt et al., 1975), and (8) becomes

\[ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa_o \nabla^2 T, \]  

(12)

where \( \kappa_o \) is the thermal diffusivity.

The validity of the Boussinesq approximation requires that the relative density perturbations produced by temperature and pressure variations be small compared with
unity and that the thickness of the convecting region be small compared with the
density scale height (e.g., Spiegel and Veronis, 1960; Schubert et al., 2001). The
thermally-induced relative density perturbations are proportional to $\alpha T'$, while the
smallness of the pressure-induced relative density perturbations requires that flow
speeds be much less than the speed of sound. The first two requirements are likely
satisfied in planetary cores, which are expected to be nearly-adiabatic with subsonic
flows (e.g., Smylie and Rochester, 1980; Guillot, 2005). However, the density within
these regions increases with depth (Guillot, 1999b), and the importance of this strat-
ification can by quantified using the density scale height, $H_\rho = -\left(\frac{1}{\rho} \frac{d\rho}{dr}\right)^{-1}$, which
gives the distance over which the density changes by a factor of $e$. The number of
density scale heights in a layer is given by $N_\rho = D/H_\rho = \ln(\rho_{bot}/\rho_{top})$, where $\rho_{bot}$
($\rho_{top}$) is the density at the bottom (top) of the region (e.g., Evonuk, 2008). These
expressions assume hydrostatic balance and a direct correlation between pressure and
density such that $\rho_{top} = \rho_{bot}e^{-D/H_\rho}$.

The Boussinesq approximation is likely reasonable for the cores of terrestrial planets
and satellites, but it is probably not adequate for the giant planets. In Earth’s core,
the density is estimated to increase by 23% across the fluid layer (Boehler, 1996), and
the layer thickness corresponds to 0.2 density scale heights. Barring compositional
gradients, the Earth is likely more strongly stratified in density than the other terres-
trial bodies due to its relatively large size. One may then argue that the Boussinesq
approximation may also be practical for the cores of all terrestrial planets and satel-
lites. We do note, however, that compositional effects may affect the validity of this assumption (e.g., Chen et al., 2008).

In contrast, density increases rapidly with depth in the giant planets. The Jovian internal structure model of Guillot (1999a) predicts the density to increase by a factor of about 100 between the $0.1 - 10^3$ GPa pressure levels. This region, thought to be important for the dynamo, extends from about the middle of the molecular envelope to near the bottom of the metallic hydrogen regime and encompasses about five density scale heights. The ice giant internal structure models of Hubbard et al. (1995) predict their convecting regions to extend at least one density scale height.

The ratio of dynamo region thickness to density scale height is of order unity or larger in the giant planets; consequently, the Boussinesq approximation is not valid. Nevertheless, we note that non-magnetic rotating convection and dynamo models of the giant planets have been studied using the Boussinesq approximation with some success (e.g., Heimpel et al., 2005; Stanley and Bloxham, 2006).

Alternatively, the anelastic approximation can be adopted (e.g., Ogura and Phillips, 1962; Glatzmaier and Gilman, 1981; Jones and Kuzanyan, 2009). This approximation is physically more realistic because it allows spatial variations in fluid properties, such as density, by linearizing the thermodynamic variables about a reference state that can vary with radius. Consequently, perturbations away from this reference state due to convective motions must be small. In addition, acoustic waves that depend on the fluid elasticity are filtered out such that the flows must also be subsonic for
the anelastic approximation to be valid. These requirements are likely satisfied in planetary interiors (e.g., Smylie and Rochester, 1980; Guillot, 2005). The anelastic governing equations are derived and discussed in detail by Braginsky and Roberts (1995, 2007) and Schubert et al. (2001).

### 2.1.2 Boundary Conditions

The dynamo equations are usually solved in the spherical shell geometry. Although this approach neglects oblateness, it is generally considered appropriate to planets. Velocity boundary conditions are impenetrable and either stress-free or no-slip surfaces. No-slip boundaries are relevant to the liquid cores in terrestrial planets, while some form of slip boundaries may be more relevant to dynamos in the outer planets. Typically, free-slip boundaries are employed for simplicity (cf. Guervilly et al., 2011). Temperature boundary conditions are either isothermal or fixed heat flux. The magnetic field must satisfy the continuity of the normal component of \( \mathbf{B} \) and the continuity of the tangential component of \( \mathbf{B} \) on the spherical shell boundaries. The latter condition assumes no surface currents and no change in permeability across the boundaries. The magnetic field outside the dynamo region must therefore be determined or specified consistently with the field in the dynamo region.
2.2 Non-dimensional Parameters

Non-dimensional parameters are meaningful ways to represent the dynamics and geometry of a system. Table 2 summarizes the parameters often used in the planetary dynamo community. The relative influences of the terms in the governing equations are typically evaluated by taking their ratios and assuming typical scales for length $D$, velocity $U$, and magnetic field strength $B$. For example, the Ekman number, $E$, estimates the ratio of the viscous to the Coriolis forces: $E = (\nu \nabla^2 u)/(2\Omega \times u) \approx (\nu U/D^2)/(2\Omega U) = \nu/2\Omega D^2$. The Rayleigh number, $Ra = \alpha g \Delta T D^3/\nu \kappa$, characterizes the ratio of buoyancy to diffusion. The Prandtl numbers, $Pr = \nu/\kappa$ and $Pm = \nu/\eta$, characterize the ratio of viscous to thermal and magnetic diffusivities, respectively. The Rossby number, $Ro = U/2\Omega D$, characterizes the ratio of inertial to Coriolis forces. The Reynolds number, $Re = UD/\nu = Ro/E$, characterizes the ratio of inertial to viscous forces, and the magnetic Reynolds number, $Rm = UD/\eta = Re Pm$, characterizes the ratio of magnetic induction to magnetic diffusion. The traditional Elsasser number, $\Lambda_i = B^2/2\rho \mu_0 \eta \Omega$, is used to characterize the ratio of Lorentz to Coriolis forces. This definition makes use of Ohm’s law to parametrize the current density, $J = \sigma(E + u \times B) \sim \sigma UB$.

As shown in Figure 5, the geometry of the dynamo region is given by the ratio of the inner to outer shell radii, $\chi = R_I/R_O$. The depth of the dynamo region relative to the planet radius, $R_O/R_P$, describes the geometry of the overlying electrically insulating region. In these parameters, $\Omega$ is rotation rate, $D = R_O - R_I$ is dynamo region
thickness, $R_O$ is outer radius of the dynamo region, $R_I$ is inner radius of the dynamo region, $\nu$ is kinematic viscosity, $\kappa$ is thermal diffusivity, $\eta$ is magnetic diffusivity, $\alpha$ is thermal expansion coefficient, $g$ is gravitational acceleration, $\Delta T$ is superadiabatic temperature contrast, $\rho$ is density, $\mu_o$ is magnetic permeability of free space, $U$ is characteristic rms velocity, and $B$ is characteristic rms magnetic field strength.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Physical Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ekman Number</td>
<td>$E = \nu/2\Omega D^2$</td>
<td>Viscous / Coriolis forces</td>
</tr>
<tr>
<td>Rayleigh Number</td>
<td>$Ra = \alpha g \Delta T D^3/\nu \kappa$</td>
<td>Buoyancy / Diffusion</td>
</tr>
<tr>
<td>Prandtl Number</td>
<td>$Pr = \nu/\kappa$</td>
<td>Viscous / Thermal diffusivities</td>
</tr>
<tr>
<td>Magnetic Prandtl Number</td>
<td>$Pm = \nu/\eta$</td>
<td>Viscous / Magnetic diffusivities</td>
</tr>
<tr>
<td>Rossby Number</td>
<td>$Ro = U/2\Omega D$</td>
<td>Inertial / Coriolis forces</td>
</tr>
<tr>
<td>Reynolds Number</td>
<td>$Re = UD/\nu$</td>
<td>Inertial / Viscous forces</td>
</tr>
<tr>
<td>Magnetic Reynolds Number</td>
<td>$Rm = UD/\eta$</td>
<td>Magnetic induction / diffusion</td>
</tr>
<tr>
<td>Elsasser Number</td>
<td>$\Lambda = B^2/2\rho \mu_o \eta \Omega$</td>
<td>Lorentz / Coriolis forces</td>
</tr>
<tr>
<td>Dynamo Region Geometry</td>
<td>$\chi = R_I/R_O$</td>
<td>Dynamo region radius ratio</td>
</tr>
<tr>
<td>Insulating Region Geometry</td>
<td>$R_O/R_P$</td>
<td>Dynamo region depth</td>
</tr>
</tbody>
</table>

Table 2: Non-dimensional parameter definitions. Symbols are defined in the text.
Figure 5: Schematic of the dynamo model geometry. Convection is driven by an imposed temperature contrast, $\Delta T = T_I - T_O > 0$, and occurs in a shell of thickness $D = R_O - R_I$ that rotates at a fixed rate of $\Omega$.

In order to estimate the non-dimensional parameters, we need to make assumptions about the dynamo regions, the fluid properties, the flow speeds, and the magnetic field strengths. This is a difficult task since many of these values are poorly constrained; the best estimates are given in Table 3. Dynamo region rotation frequencies are taken from Lodders and Fegley (1998). Note that the giant planet rotation rates are assumed to coincide with the rotation rates of their magnetic fields. The magnetic rotation rate is inferred from the periodicity of radio emissions that result from charged magnetospheric particles (e.g., Carr et al., 1981; Carr et al., 1983). However, there are complications in this approach. For example, the radio emissions of Saturn have multiple periodicities and vary in time (Gurnett et al., 2007, 2009). In addition,
Helled et al. (2010) and Karkoschka (2011) use dynamical arguments to estimate the rotation period of Neptune and find periods that differ by +68 minutes and -9 minutes, respectively, from that based on Voyager 2 radio signals (Warwick et al., 1989). These differences illustrate the uncertainties in giant planet rotation rates.

Radii of the dynamo regions are taken from Williams et al. (2007) for Mercury, from Zhang and Zhang (1995) for Venus, from Dziewonski and Anderson (1981) for Earth, from Nellis (2000) and Stevenson (1982) for the gas giants, from Schubert et al. (2007) for Ganymede, from Lee et al. (2006) and Podolak et al. (1991) for Uranus, and from Lee et al. (2006) and Stevenson (1982) for Neptune. While neither $R_O$ nor $R_I$ is well-known for any planet other than the Earth, we note that the inner dynamo region radii have an especially large degree of uncertainty.

Based on the estimated physical properties of the Earth’s core given in Stacey (2007), we assume $\nu \approx 10^{-6}$ m$^2$/s, $\eta \approx 1$ m$^2$/s, and $\kappa \approx 10^{-5}$ m$^2$/s for all bodies with liquid iron cores. Molecular transport properties of metallic hydrogen are estimated to be $\nu \approx 10^{-7}$ m$^2$/s and $\kappa \approx 10^{-6}$ m$^2$/s (Stevenson and Salpeter, 1977) and $\eta \approx 1$ m$^2$/s (Nellis, 2000); these values are used for the gas giants. Molecular transport properties of high-pressure water are estimated to be $\nu \approx 10^{-6}$ m$^2$/s (Abramson, 2005), $\kappa \approx 10^{-7}$ m$^2$/s (Abramson et al., 2001), and $\eta \approx 10^2$ m$^2$/s (Cavazzoni et al., 1999; Lee et al., 2006); these values are used for the ice giants.
**Table 3:** Summary of properties of the planets’ dynamo regions used to calculate the non-dimensional parameters given in Table 4. $R_P$ is planet radius given in Table 1. Molecular transport properties are assumed. References are given in the text.

The characteristic rms magnetic field strengths at the tops of the dynamo regions are calculated by downward continuing the measured radial magnetic fields using equation (2). This provides a lower bound for the Elsasser number since the actual magnetic field strength will likely be stronger than the estimated value due to the unmeasureable field components. In addition, the Elsasser number will be larger still within the dynamo region since toroidal field components are also present.

Dynamo action occurs only when the fluid velocities are fast enough that magnetic induction overcomes magnetic diffusion; thus, the magnetic Reynolds number must
exceed its critical value: $Rm \gtrsim 10^2$ (Roberts, 1972; Christensen and Aubert, 2006). Using this criterion, we infer the minimum flow speeds in terms of the Reynolds and Rossby numbers, given by $Re = Rm/Pm$ and $Ro = Re/E$, respectively. In the Earth’s core, secular variations suggest core speeds of $U \approx 0.5$ mm/s (Bloxham and Jackson, 1991), yielding $Rm \approx 10^3$. Assuming this value is valid for all of the planets, the flow speeds may be an order of magnitude larger than the lower bound estimates given in Table 4 that assume $Rm = 100$.

It is difficult to determine a given planet’s superadiabatic heat flux through the dynamo region, and thus, the Rayleigh number. For the Earth’s core, estimates range between $10^{22} < Ra < 10^{30}$ (Jones, 2000; Gubbins, 2001). Additional complications arise since compositional buoyancy should also be considered. Given this uncertainty, we do not make $Ra$ estimates for the other planets.

The dimensionless parameters, given in Table 4, indicate that a range of dynamo region geometries and convective dynamics are possible. The geometries can range from a sphere for Ganymede ($\chi = R_I/R_O = 0$) to a thick spherical shell for Earth ($\chi = 0.35$) to possibly a thin spherical shell for Mercury ($\chi = 0.6$). Ganymede also has the most deep-seated dynamo ($R_O/R_P = 0.75$), which may explain the satellite’s strongly-dipolar surface magnetic field.

Interestingly, Uranus and Neptune appear to have different dynamo region geometries ($\chi = 0.6$ and $\chi = 0.4$, respectively). Recent thermal evolution models suggest that this difference occurs because Neptune is fully convecting, while Uranus may only be
convecting in a thin layer (Fortney et al., 2011). Both of the gas giants have relatively thick spherical shell geometries (χ ≤ 0.5), assuming a central core is present. Further, the dynamo region of Jupiter extends up to a relatively shallow depth (R_O/R_P = 0.95). This occurs because the electrical conductivity varies strongly with radius and dynamo action may also occur in the semi-conducting region overlying the metallic hydrogen layer (Liu et al., 2008; Stanley and Glatzmaier, 2010; Heimpel and Gomez-Perez, 2011). Implications of these dimensionless parameters for convective dynamics are discussed in the following section.

<table>
<thead>
<tr>
<th>Dynamo</th>
<th>E</th>
<th>P_T</th>
<th>P_m</th>
<th>R_m</th>
<th>R_e</th>
<th>R_o</th>
<th>A_i</th>
<th>χ</th>
<th>R_O/R_P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>10^{-12}</td>
<td>0.1</td>
<td>10^{-6}</td>
<td>10^2</td>
<td>10^8</td>
<td>10^{-4}</td>
<td>10^{-5}</td>
<td>0.6</td>
<td>0.75</td>
</tr>
<tr>
<td>Venus</td>
<td>10^{-13}</td>
<td>0.1</td>
<td>&lt;10^{-2}</td>
<td>&lt;10^8</td>
<td>&lt;10^{-5}</td>
<td>-</td>
<td>0.4</td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td>Earth</td>
<td>10^{-15}</td>
<td>0.1</td>
<td>10^{-6}</td>
<td>10^2</td>
<td>10^8</td>
<td>10^{-7}</td>
<td>0.1</td>
<td>0.35</td>
<td>0.55</td>
</tr>
<tr>
<td>Ganymede</td>
<td>10^{-13}</td>
<td>0.1</td>
<td>10^{-6}</td>
<td>10^2</td>
<td>10^8</td>
<td>10^{-5}</td>
<td>10^{-3}</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>Jupiter</td>
<td>10^{-19}</td>
<td>0.1</td>
<td>10^{-7}</td>
<td>10^2</td>
<td>10^9</td>
<td>10^{-10}</td>
<td>1</td>
<td>0.2</td>
<td>0.95</td>
</tr>
<tr>
<td>Saturn</td>
<td>10^{-18}</td>
<td>0.1</td>
<td>10^{-7}</td>
<td>10^2</td>
<td>10^9</td>
<td>10^{-9}</td>
<td>10^{-2}</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Uranus</td>
<td>10^{-16}</td>
<td>10</td>
<td>10^{-8}</td>
<td>10^2</td>
<td>10^{10}</td>
<td>10^{-6}</td>
<td>10^{-4}</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>Neptune</td>
<td>10^{-16}</td>
<td>10</td>
<td>10^{-8}</td>
<td>10^2</td>
<td>10^{10}</td>
<td>10^{-6}</td>
<td>10^{-4}</td>
<td>0.4</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 4: Order of magnitude estimates of key dimensionless parameters for the planets’ dynamo regions. R_m = 10^2 is assumed to estimate lower bounds for Re = R_m/P_m and Ro = ReE. The Λ_i values, which assume magnetic field strengths at the top of the dynamo region, R_O, neglect contributions from the toroidal field and the unresolved poloidal components, which likely increase the estimate by an order of magnitude.
2.3 Fundamentals

Planetary interiors have Ekman and Rossby numbers that are much less than unity. The low values of these parameters indicate that the Coriolis force is stronger than the viscous and inertial forces, respectively. If we neglect body forces in the momentum equation (11), it becomes

\[
\frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} = -\nabla \Pi + \nu_s \nabla^2 \mathbf{u}.
\]  

(13)

When the inertial and viscous forces are weak compared to the Coriolis force, the system is in geostrophic balance:

\[
2\Omega \times \mathbf{u} = -\nabla \Pi.
\]  

(14)

This balance requires the fluid motions to follow lines of constant pressure. Further insight can be obtained by taking the curl of equation (14) and using the continuity equation (10) to obtain

\[
\Omega \cdot \nabla \mathbf{u} = 0.
\]  

(15)

If we assume \( \Omega = \Omega \hat{z} \) where \( \Omega \) is constant, this equation simplifies to

\[
\frac{\partial \mathbf{u}}{\partial z} = 0.
\]  

(16)

This is the Taylor-Proudman theorem and states that the fluid motion will be invariant along the direction of the rotation axis. However, in order for convection to occur, the Taylor-Proudman constraint cannot strictly hold as there must be slight
deviations from two-dimensionality, at least in the boundary layers, to allow overturning motions in the fluid layer (e.g., Zhang, 1992; Olson et al., 1999; Grooms et al., 2010).

Furthermore, linear asymptotic analyses predict that the azimuthal wavenumber of these columns varies as \( m = \mathcal{O}(E^{-1/3}D) \) as \( E \to 0 \) (Roberts, 1968; Jones et al., 2000; Dormy et al., 2004). Thus, in rapidly rotating systems such as planetary cores (where \( E \lesssim 10^{-12} \)), it is predicted that “quasigeostrophic” convection occurs as tall, thin columns (e.g., Kageyama et al., 2008).

In planetary dynamos, however, magnetic fields are also thought to play an important dynamical role in the convection and zonal flows. It is typically argued that the influence of magnetic fields will be important when \( \Lambda_i \gtrsim 1 \) (e.g., Zhang, 1995). In the presence of strong imposed magnetic fields and rotation, the dominant force balance is magnetostrophic - a balance between the Coriolis, pressure gradient, and Lorentz terms:

\[
2\Omega \times \mathbf{u} = -\nabla \Pi + \frac{1}{\rho_0} \mathbf{J} \times \mathbf{B}.
\] (17)

Studies of linear magnetoconvection show that the azimuthal wavenumber of convection decreases to \( m = \mathcal{O}(D) \) when a strong magnetic field \( (\Lambda_i \gtrsim \mathcal{O}(1)) \) is imposed in the limit \( E \to 0 \) (Chandrasekhar, 1961; Eltayeb and Roberts, 1970; Fearn and Proctor, 1983; Cardin and Olson, 1995). This behavior occurs because magnetic fields can relax the Taylor-Proudman constraint, allowing global-scale motions that differ fundamentally from the small-scale axial columns typical of non-magnetic, rapidly
rotating convection.

Magnetic field generation mechanisms are often described using mean field dynamo theory. This approach calculates the evolution of the large-scale, mean magnetic field, $\overline{B}$, by parameterizing the effects of small-scale, fluctuating flow and magnetic fields. The mean field induction equation is:

$$\frac{DB}{Dt} = (\overline{B} \cdot \nabla)\overline{u} + \nabla \times (\alpha_H \overline{B}) + (\eta_o + \beta) \nabla^2 \overline{B}. \quad (18)$$

The reader is referred to recent reviews by Solanki et al. (2006), Roberts (2007), and Rädler (2007) for the derivation of this equation and a discussion of the underlying assumptions. Each term is described in the following paragraphs.

The first term on the right generates magnetic fields through shearing motions as illustrated in Figure 6a. In the high electrical conductivity limit, magnetic field lines are frozen to the fluid (Alfvén, 1943). As a result, when a system exhibits differential rotation, an initially poloidal magnetic field can be sheared by the zonal flow to create a toroidal magnetic field component. This process is known as the $\Omega$-effect.

The second term on the right generates magnetic fields perpendicular to the emf $\alpha_H \overline{B}$. Typically, $\alpha_H$ is assumed proportional to the helicity (e.g., Pouquet et al., 1976; Brandenburg and Subramanian, 2005). Helicity, $H = u \cdot \omega$, arises due to correlations between velocity and vorticity fields, where vorticity $\omega = \nabla \times u$ describes the local rotational motions of the fluid. Magnetic field generation through helical fluid motions is illustrated in Figure 6b. Here, a parcel of upwelling fluid deforms an
initially toroidal magnetic field line, which is simultaneously twisted out of the plane by the rotational $\omega$ motion. This flux loop may then detach if magnetic diffusion is large enough for reconnection to occur, creating a magnetic field component that is perpendicular to the original field line orientation. This process is known as the $\alpha$-effect. It is presently unclear, however, how to parameterize $\alpha_H$ without explicitly calculating the helicity (e.g., Proctor, 2003; Brown et al., 2010).

The third term on the right describes molecular and turbulent magnetic diffusion of the mean magnetic field. Here, $\eta_o$ is the molecular magnetic diffusivity and $\eta_t$ is the (isotropic) turbulent magnetic diffusivity. Turbulent eddies then enhance the magnetic dissipation of the system. Starchenko and Jones (2002) estimate $\eta_t \sim 10^3$ m$^2$/s for the gas giants and $\eta_t \sim 10$ m$^2$/s for the Earth’s core. These values are about 1000 and 10 times larger, respectively, than the estimated molecular magnetic diffusivities of these planets (Table 3).

With the fundamentals of convection and dynamo action having been reviewed, we will next discuss how these processes are modeled.
Figure 6: Illustration of the a) $\Omega$-effect and b) $\alpha$-effect. The $\Omega$-effect creates a toroidal magnetic field component from an initially poloidal magnetic field through zonal shearing motions. The $\alpha$-effect creates a poloidal magnetic field component from an initially toroidal magnetic field through helical fluid motions. Figure adapted from Roberts (2007).
2.4 Simulating Convection and Dynamo Action

2.4.1 Numerical Model

For my dissertation work, I am using the community planetary dynamo code MagIC version 3.44 (Wicht, 2002; Christensen and Wicht, 2007). This model simulates three-dimensional, time-dependent thermal convection of a Boussinesq fluid in a spherical shell rotating with constant angular velocity $\Omega \hat{z}$. Two types of simulations are considered: 

1) non-magnetic rotating convection models which employ an electrically-insulating fluid and
2) dynamo models which employ an electrically-conducting fluid.

The shell geometry is defined by the ratio of the inner to outer shell radii $\chi = R_I / R_O$. The shell boundaries are isothermal with an imposed (superadiabatic) temperature contrast $\Delta T$ between the inner and outer boundaries. The mechanical boundary conditions are impenetrable and either no-slip or stress-free. Gravity varies linearly with spherical radius. The region exterior to the fluid shell is electrically insulating, and the electrical conductivity of the rigid inner sphere is chosen to be the same as that of the convecting fluid region.

The dimensionless governing equations for this system are

\[ E \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla^2 \mathbf{u} \right) + \hat{z} \times \mathbf{u} + \frac{1}{2} \nabla \Pi = \frac{r}{R} \frac{r}{R_O} T + \frac{1}{2P_m} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (19) \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{P_m} \nabla^2 \mathbf{B}, \quad (20) \]

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T, \quad (21) \]
\[ \nabla \cdot \mathbf{u} = 0, \nabla \cdot \mathbf{B} = 0, \]  

(22)

where \( \mathbf{u} \) is the velocity vector, \( \mathbf{B} \) is the magnetic induction, \( T \) is the temperature, and \( \Pi \) is the non-hydrostatic pressure. We make use of typical non-dimensionalizations used in the planetary dynamo literature: shell thickness \( D = R_O - R_I \) as length scale; \( \Delta T \) as temperature scale; \( \tau_\nu \sim D^2/\nu \) as time scale; \( \rho \nu \Omega \) as pressure scale; \( \nu/D \) as velocity scale such that the non-dimensional globally-averaged rms flow velocity is equal to the Reynolds number \( Re = UD/\nu \); and \( \sqrt{2\rho\mu_o\eta\Omega} \) as magnetic induction scale such that the square of the non-dimensional globally-averaged rms magnetic field strength is equal to the traditional Elsasser number \( \Lambda_i = B^2/2\rho\mu_o\eta\Omega \). In these definitions, \( \rho \) is density, \( \nu \) is kinematic viscosity, \( \kappa \) is thermal diffusivity, \( \eta \) is magnetic diffusivity, \( \mu_o \) is magnetic permeability of free space, and \( \alpha \) is thermal expansion coefficient.

The non-dimensional control parameters are: the shell geometry \( \chi = R_I/R_O \), the Rayleigh number \( Ra = \alpha g \Delta T D^3/\nu \kappa \), the Ekman number \( E = \nu/2\Omega D^2 \), the Prandtl number \( Pr = \nu/\kappa \), and the magnetic Prandtl number \( Pm = \nu/\eta \). The control parameter definitions are summarized in Table 2; the diagnostic parameters are defined in Table 5.

Hyperdiffusion may be employed to increase numerical stability by damping the small-scale components of the velocity, thermal, and magnetic fields (cf. Grote et al., 2000b; Zhang et al., 1998; Zhang and Schubert, 2000). The viscous, thermal, and magnetic
diffusivities are multiplied by a factor

\[ d(l) = 1 + A_{HD} \left( \frac{l + 1 - l_{HD}}{l_{\text{max}} + 1 - l_{HD}} \right)^\beta, \]  

(23)

where \( A_{HD} \) is the hyperdiffusion amplitude, \( l \) is the spherical harmonic degree, \( l_{HD} \) is the degree above which hyperdiffusion starts to act, \( l_{\text{max}} \) is the maximum harmonic degree, and \( \beta \) is the exponent of hyperdiffusion.

### 2.4.2 Numerical Method

MagIC solves equations (19) – (22) simultaneously for \( u, B, \) and \( T \) using the pseudospectral method, as first developed by Glatzmaier (1984) and subsequently modified by U. Christensen and J. Wicht (Christensen et al., 1999; Wicht, 2002; Christensen and Wicht, 2007). The velocity and magnetic induction vectors are decomposed into poloidal and toroidal scalar potentials, which are expanded in Chebyshev polynomials in the radial direction and in spherical harmonic functions on spherical surfaces.

MagIC utilizes mixed implicit and explicit time stepping. The Coriolis and nonlinear terms are treated explicitly using a second-order Adams-Bashforth scheme and the diffusion, pressure, and linear terms are treated implicitly using a Crank-Nicolson time step. The implicit time step can vary over time and is limited by a modified MHD Courant criterion which accounts for viscous and ohmic damping of short-wavelength Alfvén-type oscillations (Christensen et al., 1999). This damping allows the use of a slightly larger numerical time step compared to the unmodified MHD
Courant criterion. The code also utilizes OpenMP parallelization.

2.4.3 Model Limitations

A significant limitation of the model is the use of OpenMP parallelization since the code can only be run in parallel on machines with shared memory. This restriction limits both the availability of machines and the number of processors we can use. The simulations presented in this dissertation were carried out on the San Diego Supercomputing Center DataStar supercomputer and the NASA Ames supercomputers Schirra and Pleaides. I currently use only Pleaides since both DataStar and Schirra have been decommissioned. Further, Pleaides’ architecture of two six-core processors per node allows a maximum of 12 threads for OpenMP applications.

Towards making the code more efficient, Dr. Johannes Wicht is working with Dr. Raffaele Montuoro, a computational scientist at the Texas A & M University Supercomputing Facility, to develop a new version of the code that instead utilizes hybrid OpenMP/MPI parallelization. This alternate method is superior because it allows the use of distributed memory machines and enables the use of multiple nodes.

Such technological limitations also dictate that a significant disparity exists between some of the simulated control parameter values and those expected for planetary settings. In particular, we must use artificially large viscous diffusivities in order to damp the small scale motions that cannot be resolved. Consequently, the Ekman number
(viscous/Coriolis force ratio) and the magnetic Prandtl number (viscous/magnetic diffusivity ratio) are too large by many orders of magnitude. Furthermore, the large viscosities of the models also require superadiabatic temperature gradients that are larger than those expected to occur in planets in order to excite convection and drive dynamo action. All planetary dynamo models are overdriven in this way (Christensen and Wicht, 2007).

Because of this dichotomy between simulated and planetary parameter values, it is important to determine what aspects of numerical models can be extrapolated to planetary settings. This goal is especially relevant to the simulations presented in Chapter 3, where we argue that the dynamic Elsasser number, \( \Lambda_d \), should estimate the Lorentz to Coriolis force ratio in both models and planets, while the generation of dipolar magnetic fields through viscously-stabilized helices may only be applicable to dynamo models with moderate Ekman numbers.

Numerical dynamo models do achieve realistic values of some dimensionless parameters. An important example is the magnetic Reynolds number, which estimates the ratio of magnetic induction to magnetic diffusion. This parameter is thought to range between 100–1000 for both planets and the majority of dynamo models (e.g., Starchenko and Jones, 2002; Christensen and Aubert, 2006). It is often argued that models are able to generate magnetic fields that exhibit behaviors similar to those of planets because both systems have similar levels of magnetic induction (e.g., Christensen and Wicht, 2007). This assumption is tested in Chapter 5.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{E}_K )</td>
<td>( \frac{1}{2V} \int \mathbf{u} \cdot \mathbf{u} , dV )</td>
<td>Kinetic energy density</td>
</tr>
<tr>
<td>( \mathcal{E}_{K}^{\text{Axisymmetric}} )</td>
<td>( \mathcal{E}<em>K - \mathcal{E}</em>{K,Toroidal} )</td>
<td>Convective kinetic energy density</td>
</tr>
<tr>
<td>( \mathcal{E}_M )</td>
<td>( \frac{1}{2V} \int \mathbf{B} \cdot \mathbf{B} , dV )</td>
<td>Magnetic energy density</td>
</tr>
<tr>
<td>( \overline{l_u} )</td>
<td>( \sum l(u_i \cdot u_i)/2\mathcal{E}_K )</td>
<td>Characteristic degree of the flow</td>
</tr>
<tr>
<td>( \overline{l_B} )</td>
<td>( \sum l(B_i \cdot B_i)/2\mathcal{E}_M )</td>
<td>Characteristic degree of the B field</td>
</tr>
<tr>
<td>( \overline{m_u} )</td>
<td>( \sum m(u_m \cdot u_m)/2\mathcal{E}_K )</td>
<td>Characteristic order of the flow</td>
</tr>
<tr>
<td>( \overline{m_B} )</td>
<td>( \sum m(B_m \cdot B_m)/2\mathcal{E}_M )</td>
<td>Characteristic order of the B field</td>
</tr>
<tr>
<td>( k_u )</td>
<td>( \sqrt{\overline{l_u}^2 + \overline{m_u}^2} )</td>
<td>Characteristic wavenumber of the flow</td>
</tr>
<tr>
<td>( k_B )</td>
<td>( \sqrt{\overline{l_B}^2 + \overline{m_B}^2} )</td>
<td>Characteristic wavenumber of the B field</td>
</tr>
<tr>
<td>( C_{\omega z} )</td>
<td>( \frac{\sum_{s,s}</td>
<td>(\omega' \cdot \hat{\mathbf{z}})</td>
</tr>
<tr>
<td>( H_{\text{rel}}^{\text{z}} )</td>
<td>( \frac{\langle H_z \rangle_h}{\langle (u_z u_z) \rangle_h (\langle \omega_z \omega_z \rangle_h)^{1/2}} )</td>
<td>Relative axial helicity</td>
</tr>
<tr>
<td>Nusselt number</td>
<td>( Nu = \frac{R_Q}{R_f} \frac{qD}{\rho C_p \kappa \Delta T} )</td>
<td>Total Heat Transfer</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>( Re = \frac{UL}{\nu} = \sqrt{2E_K} )</td>
<td>Conductive Heat Transfer</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Inertial force</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Viscous force</td>
</tr>
</tbody>
</table>
## Table 5: Summary of non-dimensional diagnostic parameters. Symbols are defined in the text. In these definitions, $\ell_B = (\pi D/2)/k_B$ is assumed to be the characteristic quarter-wavelength of the magnetic field.
3 Survey of Terrestrial-style Dynamo Models

3.1 Introduction

Terrestrial-style dynamos are characterized by the dynamo source region being surrounded by solid layers such that no-slip boundary conditions are appropriate. While the geodynamo is the most-studied terrestrial-style dynamo, the processes that control the magnetic field strength, morphology, and secular variation are still not known. Even less is known about the processes that govern the magnetic fields of Mercury and Ganymede and were responsible for the now extinct fields of the Moon and Mars. In order to understand when and why a particular style of magnetic behavior will occur, it is important to understand the coupling between heat transfer, fluid flows, and magnetic fields across parameter space.

Towards this end, I have carried out an extensive survey to examine the behavior of terrestrial-style dynamos, detailed in Section 3.2. Convective heat transfer is examined in Section 3.3, and Section 3.4 quantifies the influence of the magnetic field on convection dynamics and investigates the transition from dipolar to multipolar magnetic field generation. The convective (non-zonal) flow structures are found to be largely invariant of the presence of magnetic fields in our models, contrary to predictions by linear magnetoconvection theory. We also show that the Elsasser number traditionally used to measured the relative strengths of the Lorentz and Coriolis forces does not accurately predict this force balance and, instead, advocate the use of an
alternatively defined ‘dynamic Elsasser number’. In addition, we put forward a new hypothesis to explain the dipole to multipole transition – a breakdown of helical flow that occurs when inertial forces exceed the viscous diffusion forces – which implies that present day dynamo models with moderate rotation rates \( (E \gtrsim 10^{-4}) \) are too viscous to reproduce the physical mechanisms of field generation in planetary interiors correctly where viscosity is negligible.

### 3.2 Numerical Model

The terrestrial-style survey consists of 19 planetary dynamo models and 16 non-magnetic rotating convection models with no-slip boundary conditions. All of the control parameters are fixed, except for the Rayleigh number, to values that are commonly used in the present day planetary dynamo literature: \( \chi = 0.4, E = 10^{-4}, Pr = 1, \) and \( Pm = [0, 2] \). The value of \( Pm \) is set to 2 in the dynamo models such that the magnetic Reynolds number \( Rm = RePm \gtrsim 10^2 \), a necessary condition for dynamo action. The Rayleigh number is varied from \( Ra = 1.42 \times 10^6 \) to \( Ra = 8.48 \times 10^8 \). The critical Rayleigh number, \( Ra_c \), denotes the onset of convection. Here, we use the inferred scaling \( Ra_c = 3.5E^{-4/3} \) from King et al. (2010), giving \( Ra_c = 7.5 \times 10^5 \) for our simulations. The Rayleigh numbers then fall in the range \( 1.9Ra_c \leq Ra \leq 1125Ra_c \). This dataset is among the broadest surveys of supercriticality in the literature. Furthermore, this \( Ra_c \) estimate is similar to the value found for non-magnetic rotating convection in the same geometry (Al-Shamali et al., 2004). This
raises the questions: Do dynamo-generated magnetic fields fundamentally alter the convection dynamics as predicted by linear theory? The control parameter definitions are summarized in Table 2, the diagnostic parameters are defined in Table 5, and the dataset is detailed in Table 6.

The largest numerical grid uses 192 spherical harmonic modes, 65 radial levels in the outer shell, and 17 radial levels in the inner core. No azimuthal symmetries are employed. Dynamo models are initialized using the results of prior dynamo simulations. Non-magnetic models are initialized by turning off the magnetic field of the associated dynamo model, similar to Zhang et al. (1998) and Sarson and Jones (1998). Comparing results obtained with different initial conditions for a limited number of cases showed no significant differences in time-averaged behaviors. Once the initial transient behavior has subsided, cases with \( Ra \leq 11Ra_c \) are time averaged over at least three viscous diffusion times (\( \sim 2400 \) rotations), cases with \( 11Ra_c < Ra \leq 56Ra_c \) are averaged over at least 0.5 viscous diffusion times (\( \sim 400 \) rotations), and all other cases are averaged over at least 0.04 viscous diffusion times (\( \sim 30 \) rotations).
| $Ra$   | $\frac{Ra}{Ra_{c}}$ | $N_{ro}$ | $l_{max}$ | $Nu$ | $Re_{c}$ | $Re$ | $C_{\omega z}$ | $|H_{z}^{\text{rel}}|$ | $\overline{k_{u}}$ | $\overline{k_{B}}$ | $f$ | $\Lambda_{i}$ | $\Lambda_{d}$ | $\lambda$ | $\frac{F_{L}}{F_{C}}$ |
|-------|----------------------|---------|----------|------|---------|------|----------------|----------------|----------------|----------------|----|-------------|-------------|------|----------------|
| $1.42 \times 10^{6}$ | 1.9 | 41 | 64 | 1.88 | 32 | 34 | 0.691 | 0.493 | 15.0 | 11.6 | 0.89 | 1.27 | 0.139 | 0.059 | 0.153 |
| | | | | | | | | | | (1.74) | (41) | (43) | (0.700) | (0.535) | (17.5) |
| $2.12 \times 10^{6}$ | 2.8 | 41 | 64 | 2.54 | 51 | 52 | 0.661 | 0.475 | 17.3 | 13.6 | 0.76 | 2.11 | 0.175 | 0.089 | 0.200 |
| | | | | | | | | | (2.32) | (64) | (67) | (0.670) | (0.511) | (19.7) |
| $2.83 \times 10^{6}$ | 3.8 | 41 | 64 | 3.19 | 70 | 72 | 0.600 | 0.407 | 18.6 | 15.8 | 0.65 | 2.43 | 0.171 | 0.111 | 0.222 |
| | | | | | | | | | (2.98) | (89) | (92) | (0.631) | (0.492) | (19.4) |
| $3.54 \times 10^{6}$ | 4.7 | 41 | 64 | 3.78 | 90 | 91 | 0.601 | 0.367 | 19.5 | 18.1 | 0.57 | 2.34 | 0.147 | 0.124 | 0.253 |
| | | | | | | | | | (3.74) | (113) | (118) | (0.603) | (0.383) | (18.9) |
| $3.68 \times 10^{6}$ | 4.9 | 41 | 64 | 3.88 | 93 | 95 | 0.597 | 0.358 | 19.6 | 18.7 | 0.56 | 2.36 | 0.147 | 0.129 | 0.225 |
| | | | | | | | | | (3.88) | (117) | (123) | (0.606) | (0.347) | (18.9) |
| $3.75 \times 10^{6}$ | 5.0 | 41 | 64 | 3.94 | 96 | 98 | 0.608 | 0.350 | 19.7 | 18.9 | 0.55 | 2.28 | 0.141 | 0.129 | 0.180 |
| | | | | | | | | | (3.94) | (119) | (125) | (0.625) | (0.372) | (18.8) |
| $3.82 \times 10^{6}$ | 5.1 | 41 | 64 | 4.03 | 119 | 124 | 0.631 | 0.366 | 19.1 | 26.9 | 0.21 | 0.16 | 0.011 | 0.049 | 0.018 |
| | | | | | | | | | (4.00) | (121) | (127) | (0.611) | (0.394) | (18.8) |
| $Ra$   | $\frac{Ra}{Ra_c}$ | $N_{ro}$ | $l_{max}$ | $Nu$ | $Re_c$ | $Re$ | $C_{wz}$ | $|H^z_{rel}|$ | $\overline{k_u}$ | $\overline{k_B}$ | $f$ | $\Lambda_i$ | $\Lambda_d$ | $\lambda$ | $\frac{F_i}{F_C}$ |
|--------|------------------|----------|-----------|------|--------|------|----------|-------------|----------------|----------------|----|------------|------------|--------|-------------|
| $3.96 \times 10^6$ | 5.3 | 41 | 64 | 4.15 | 123 | 128 | 0.586 | 0.330 | 19.1 | 26.7 | 0.18 | 0.18 | 0.012 | 0.052 | 0.017 |
| | | | | | | | | (4.13) | (126) | (132) | (0.607) | (0.349) | (18.7) |
| $4.11 \times 10^6$ | 5.5 | 41 | 64 | 4.28 | 127 | 131 | 0.589 | 0.334 | 19.0 | 26.8 | 0.18 | 0.22 | 0.015 | 0.057 | 0.019 |
| | | | | | | | | (4.25) | (130) | (136) | (0.588) | (0.302) | (18.7) |
| $4.24 \times 10^6$ | 5.6 | 41 | 85 | 4.38 | 130 | 135 | 0.599 | 0.275 | 19.0 | 27.2 | 0.20 | 0.24 | 0.015 | 0.059 | 0.017 |
| | | | | | | | | (4.36) | (133) | (140) | (0.583) | (0.297) | (18.6) |
| $6.00 \times 10^6$ | 8.0 | 41 | 85 | 5.57 | 169 | 174 | 0.570 | 0.193 | 19.2 | 29.0 | 0.12 | 0.68 | 0.036 | 0.108 | 0.048 |
| | | | | | | | | (5.57) | (179) | (188) | (0.546) | (0.176) | (18.1) |
| $8.50 \times 10^6$ | 11 | 41 | 128 | 6.72 | 210 | 216 | 0.541 | 0.148 | 19.1 | 31.4 | 0.13 | 1.54 | 0.072 | 0.176 | 0.117 |
| | | | | | | | | (6.77) | (231) | (246) | (0.519) | (0.144) | (17.7) |
| $1.42 \times 10^7$ | 19 | 49 | 128 | 8.39 | 285 | 295 | 0.454 | 0.095 | 18.7 | 34.6 | 0.06 | 3.30 | 0.123 | 0.283 | 0.203 |
| | | | | | | | | (8.52) | (326) | (353) | (0.429) | (0.121) | (17.2) |
| $2.10 \times 10^7$ | 28 | 49 | 128 | 9.73 | 357 | 368 | 0.395 | 0.101 | 18.6 | 37.8 | 0.06 | 5.49 | 0.180 | 0.399 | 0.280 |
| | | | | | | | | (9.91) | (417) | (453) | (0.406) | (0.120) | (17.0) |
Table 6: Input and output parameters for fixed $\chi = 0.4$, $E = 10^{-4}$, $Pr = 1$, and $Pm = [0, 2]$. Non-magnetic values are given in parentheses.

The number of radial grid points in the outer and inner cores is denoted $N_{ro}$ and $N_{ri} = 17$, respectively. All other parameters are defined in Tables 2 and 5. Asterisks indicate the use of hyperdiffusion (see Table 7). All quantities are time-averaged, except for $C_{\omega z}$, $|H_{rel}^z|$, and $F_L/F_C$, which are instantaneous. The models are in Regime I (columnar, dipolar) above the first dashed line, in Regime II (columnar, multipolar) between the dashed lines, and in Regime III (3D, multipolar) below the second dashed line.

| $Ra$          | $\frac{Ra}{Ra_c}$ | $N_{ro}$ | $l_{max}$ | $Nu$ | $Re_c$ | $Re$ | $C_{\omega z}$ | $|H_{rel}^z|$ | $\overline{k_u}$ | $\overline{k_B}$ | $f$ | $\Lambda_i$ | $\Lambda_d$ | $\lambda$ | $\frac{F_L}{F_C}$ |
|---------------|-------------------|---------|----------|------|--------|------|----------------|---------------|---------------|---------------|----|-----------|-----------|--------|-----------------|
| $2.10 \times 10^7$ | 28 | 49 | 128 | 9.71 | 352 | 358 | 0.432 | 0.071 | 15.6 | 22.7 | 0.08 | 7.56 | 0.153 | 0.385 | 0.070 |
| $4.24 \times 10^7$ | 56 | 49 | 192 | 12.4 | 540 | 557 | 0.343 | 0.101 | 17.3 | 44.9 | 0.04 | 11.6 | 0.297 | 0.687 | 0.408 |
|                |      |       |      |      |      |      |         |         |      |       |      |      |        |       |       |
| $(12.7)$ | $(652)$ | $(686)$ | $(0.313)$ | $(0.125)$ | $(17.4)$ |
| $8.00 \times 10^7$ | 106 | 49 | 192 | 15.2 | 821 | 846 | 0.294 | 0.086 | 15.7 | 51.0 | 0.03 | 20.8 | 0.399 | 1.05 | 0.505 |
| $1.42 \times 10^8$ | 188 | 65 | 192 | 17.8 | 1116 | 1156 | 0.263 | 0.086 | 14.5 | 58.1 | 0.04 | 36.5 | 0.584 | 1.58 | 0.702 |
| $4.24 \times 10^8$ | 562 | 65 | 192 | 23.4 | 1790 | 2014 | 0.255 | 0.047 | 12.5 | 50.5 | 0.05 | 106 | 0.844 | 2.34 | 1.27 |
|                |      |       |      |      |      |      |         |         |      |       |      |      |        |       |       |
| $(25.0)$ | $(2139)$ | $(2470)$ | $(0.225)$ | $(0.045)$ | $(13.7)$ |
| $8.48 \times 10^8$ | 1125 | 65 | 192 | 27.0 | 2281 | 2618 | 0.235 | 0.030 | 13.2 | 46.3 | 0.02 | 196 | 1.11 | 2.92 | 1.60 |
Hyperdiffusion is used in our most supercritical models to increase numerical stability. As shown in Table 7, the strongest form of hyperdiffusion used in this study has $A_{HD} = 10$, $l_{HD} = 20$, $\beta = 2$, and $l_{max} = 192$. The harmonic degree above which hyperdiffusion applies, $l_{HD}$, was chosen to be greater than the characteristic harmonic degree of the flows, $l_u$. By comparing a hyperdiffusive model at $Ra = 2.10 \times 10^7$ to an otherwise identical non-hyperdiffusive model, we have determined that this hyperdiffusion does not significantly affect the flow speeds or heat transfer. The Reynolds and Nusselt numbers differ by less than 3%. However, the characteristic harmonic wavenumber of the flow decreases by 16% in the hyperdiffusive model, and the Elsasser number and dipolarity both increase by about 35%. Thus, the use of hyperdiffusion generally leads to broader scale flows (cf. Zhang and Schubert, 2000) and strengthens the dipole component of the magnetic field.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$l_{max}$</th>
<th>$A_{HD}$</th>
<th>$l_{HD}$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.10 \times 10^7$</td>
<td>128</td>
<td>10</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>$4.24 \times 10^8$</td>
<td>192</td>
<td>5</td>
<td>50</td>
<td>2</td>
</tr>
<tr>
<td>$8.48 \times 10^8$</td>
<td>192</td>
<td>10</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 7: Summary of the hyperdiffusion parameters used in the terrestrial survey. Symbols are defined in equation 23.
3.3 Convective Heat Transfer

In King et al. (2010), we examine convective heat transfer and show that its behavior in terrestrial-style dynamo models is similar to that found in the non-magnetic, planar laboratory and numerical experiments of King et al. (2009). Two heat transfer regimes are found in both studies, and King et al. (2009) argue that the transition is governed by boundary layer dynamics, particularly by the relative thicknesses of the thermal boundary layer (non-rotating) and the mechanical Ekman boundary layer (rotating). Their results predict that rotation will control the dynamics when the Ekman boundary layer is thinner than the thermal boundary layer and the convection will be inertially-dominated when the thermal boundary layer is thinner than the Ekman boundary layer.

Theoretical scalings of the boundary layer thicknesses permit a prediction of the transition point between the two regimes. The Ekman and thermal boundary layers have thicknesses:

\[ \delta_E \sim E^{1/2} \] and \[ \delta_\kappa \sim Nu^{-1} \],

(24)

respectively (Greenspan, 1968; Spiegel, 1971). The transition is predicted to occur where the boundary layers cross; i.e. when \( \delta_E = \delta_\kappa \). The Nusselt number at which the transition should occur is then

\[ Nu_t \sim E^{-1/2}. \]

(25)

The Nusselt number can be related to the Rayleigh number using the heat transfer
scaling law $Nu \sim Ra^{2/7}$, which is typically found in weakly rotating systems (e.g., Castaing et al., 1989; Glazier et al., 1999). This allows us to solve for the Rayleigh number at which the boundary layers are predicted to cross:

$$Ra_t = E^{-7/4},$$

which we call the transitional Rayleigh number. Consequently, the heat transfer should follow the non-rotating scaling when $Ra/Ra_t \gtrsim 1$ and the rapidly-rotating scaling when $Ra/Ra_t < 1$.

Using my suite of simulations, I test the applicability of this hypothesis to terrestrial-style dynamo models by calculating the Ekman and thermal boundary layer thicknesses. Following the seminal study of Belmonte et al. (1994), the velocity boundary layer thickness is defined as the radial location of the first local maximum in rms velocity above the model’s inner boundary. Similarly, the thermal boundary layer is defined as the radial location of the first local maximum in temperature variance above the model’s inner boundary (Belmonte et al., 1994). The added complication of magnetic fields does not influence our evaluation of the viscous boundary layer thicknesses, which are calculated directly from the velocity field in each case.

These calculations show that the boundary layer thicknesses become comparable at $Ra \approx Ra_t$, consistent with the predicted transition (Figure 7). The convective heat transfer behavior, discussed below, is also shown to change across this transition.

To supplement my survey, a large compilation of planetary dynamos from the litera-
ture (e.g., Christensen and Aubert, 2006; Olson and Christensen, 2006; Christensen et al., 2009) is used to test the robustness of heat transfer scaling laws. Heat transfer efficiency is measured by the Nusselt number and is often expressed in terms of a power law scaling in the form $Nu \propto Ra^\alpha$. The Nusselt number argued to scale theoretically as $Nu = (Ra/Ra_c)^{6/5}$ with $Ra_c = 3.5E^{-4/3}$ when the convection is rapidly-rotating ($Ra/Ra_t < 1$) and empirically shown to scale as $Nu = 0.075Ra^{2/7}$ when the convection is weakly-rotating ($Ra/Ra_t \gtrsim 1$). Interestingly, these scaling exponents are also found in non-magnetic studies (e.g., Christensen, 2002; Castaing et al., 1989; Glazier et al., 1999; King et al., 2009). This study suggests that these basic behaviors occur irrespective of magnetic field strength or fluid layer geometry in present day dynamo models with moderate Ekman numbers.

Figure 7: Time-averaged boundary layer thicknesses versus the transition parameter, $Ra/Ra_t$, on the a) inner and b) outer boundaries. The Ekman boundary layer thickness is shown as downward pointing blue triangles, and the thermal boundary layer thickness is shown as upward pointing red triangles. Boundary layer thicknesses are normalized by the shell thickness, $D$. The distance between errorbars represents the local spatial resolution.
3.4 The Weak Influence of Magnetic Fields in Planetary Dynamo Models

3.4.1 Introduction

Our results in Section 3.3 imply that convective heat transfer in planetary dynamo models is largely invariant of their magnetic fields. Further, axial convective flow structures are maintained in many rotating magnetoconvection and dynamo studies despite having strong ($\Lambda_i \gtrsim O(1)$) magnetic fields which should relax the Taylor-Proudman constraint (e.g., Olson and Glatzmaier, 1995; Zhang, 1995; Kageyama and Sato, 1997; Christensen et al., 1999; Zhang and Schubert, 2000; Jones, 2007; Jault, 2008). These studies imply that the traditional force balance argument from linear analysis using the Elsasser number, $\Lambda_i$, may not be an adequate measure of the dynamical influence of the Lorentz force in convection systems. Alternate characterizations of the Lorentz to Coriolis force ratio must therefore be considered.

In this section, we contrast dynamo models with non-magnetic, but otherwise identical, rotating convection models to quantify the influence of magnetic fields on convective flow dynamics. While comparisons between dynamo and non-magnetic simulations have been conducted previously by Christensen et al. (1999), Grote and Busse (2001), and Aubert (2005), the studies are typically limited to convection less than 40 times critical and dipolar magnetic field geometries. Our survey is complementary to these earlier studies since it extends the comparison to convection more than 1000
times critical, considers both dipolar and multipolar magnetic fields, and makes no assumptions of azimuthal symmetries. We measure the strengths and structures of magnetic fields and convective flows, as well as heat transfer efficiency. These measurements show that the magnetic fields in our models do not play a dominant role in convection (non-zonal) dynamics and that a sharp transition occurs between dipolar and multipolar dynamos. Behavioral regimes found in our models are discussed in Section 3.4.2, and we analyze parameterizations of the magnetic field influence in Section 3.4.3. In Section 3.4.4, we examine the transition from dipolar to multipolar dynamos. Our conclusions are given in Section 3.4.5.

3.4.2 Behavioral Regimes

Three behavioral regimes based on magnetic field morphology and convective planform are identified in our suite of dynamo models. Models in Regime I (\( Ra \leq 5.0Ra_c \)) are characterized by columnar convection and dipole-dominated magnetic fields, while models in Regime II (\( 5.1Ra_c \leq Ra \lesssim 19Ra_c \)) have columnar convection and multipolar magnetic fields. In Regime III (\( Ra \gtrsim 19Ra_c \)), models are characterized by three-dimensional convection and multipolar magnetic fields. These regimes are discussed below.
**Magnetic Field Morphology**  The magnetic field morphology is quantified by its dipolarity:

\[
f = \left( \frac{\int B_l(r = R_O) \cdot B_l(r = R_O) \, dA}{\int B(r = R_O) \cdot B(r = R_O) \, dA} \right)^{1/2}.
\] (27)

Here, \( B(r = R_O) \) is the magnetic induction vector at the outer shell boundary, \( l = 1 \) indicates the dipolar component, and \( \int dA \) integrates over the outer spherical shell surface. The magnetic field is perfectly dipolar when \( f = 1 \). We consider cases with \( f \gtrsim 0.5 \) to be dipole-dominated. The dipolarity of Earth’s magnetic field at the core-mantle boundary up to spherical harmonic degree 12 is \( f \approx 0.68 \). Here, we calculate \( f \) using the full spectrum and note that the \( f(Ra) \) behavior is similar when the calculation of dipolarity is truncated at \( l = 12 \).

Figure 8 shows time-averaged dipolarity plotted versus the Rayleigh number. A first-order transition between dipole-dominated \( (f \gtrsim 0.55) \) and multipolar \( (f \lesssim 0.21) \) magnetic fields occurs at \( Ra = 5.1 Ra_c \). This defines the boundary between Regimes I and II. The dipolarity also decreases by about a factor of two between Regimes II \( (f \gtrsim 0.13) \) and III \( (f \lesssim 0.06) \). This transition, as will be shown below, is defined by the transition to three-dimensional convective flow structures.

Figure 9 shows the radial magnetic fields near the outer shell boundary for four dynamo models with \( Ra = 1.9 Ra_c, 4.9 Ra_c, 5.6 Ra_c, \) and \( 562 Ra_c \). These cases clearly indicate that the magnetic field transitions from dipole-dominated dynamos nearly aligned with the rotation axis to unstructured, multipolar dynamos.
Figure 8: Time-averaged dipolarity, $f$, versus the Rayleigh number. The solid vertical line at $Ra = 3.82 \times 10^6$ indicates the sharp transition between dipolar and multipolar dynamos, and the long-dashed vertical line at $Ra = 1.42 \times 10^7$ indicates where the convective flow structures are no longer axial. The three behavioral regimes are also highlighted.
<table>
<thead>
<tr>
<th>Dynamo</th>
<th>Dipolar</th>
<th>Multipolar</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Ra=1.9Ra₀</td>
<td>f=0.89</td>
<td>e) Ra=5.6Ra₀</td>
</tr>
<tr>
<td>Cₜω₀=0.69</td>
<td></td>
<td>Cₜω₀=0.60</td>
</tr>
<tr>
<td>b) Ra=4.9Ra₀</td>
<td>f=0.56</td>
<td>g) Ra=5.6Ra₀</td>
</tr>
<tr>
<td>Cₜω₀=0.60</td>
<td></td>
<td>Cₜω₀=0.26</td>
</tr>
<tr>
<td>c) Ra=5.6Ra₀</td>
<td>f=0.20</td>
<td>h) Ra=562Ra₀</td>
</tr>
<tr>
<td>Cₜω₀=0.58</td>
<td></td>
<td>Cₜω₀=0.23</td>
</tr>
<tr>
<td>d) Ra=562Ra₀</td>
<td>f=0.05</td>
<td></td>
</tr>
<tr>
<td>Cₜω₀=0.26</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Non-magnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>e) Ra=1.9Ra₀</td>
</tr>
<tr>
<td>Cₜω₀=0.70</td>
</tr>
<tr>
<td>f) Ra=4.9Ra₀</td>
</tr>
<tr>
<td>Cₜω₀=0.61</td>
</tr>
<tr>
<td>g) Ra=5.6Ra₀</td>
</tr>
<tr>
<td>Cₜω₀=0.58</td>
</tr>
<tr>
<td>h) Ra=562Ra₀</td>
</tr>
<tr>
<td>Cₜω₀=0.23</td>
</tr>
</tbody>
</table>

**Columnar** | **Three-dimensional**

Figure 9: Instantaneous radial magnetic fields near the outer shell boundary (top row) and isosurfaces of axial vorticity for select dynamo (middle row) and non-magnetic (bottom row) models. Purple (green) indicates radially outward (inward) directed magnetic fields. Red (blue) indicates cyclonic (anticyclonic) vorticity. Each subplot has its own colorscale. The inner yellow sphere represents the inner core. The outer boundary layer has been excluded for clarity. Below each image is the dipolarity, \( f \), and axial vorticity columnarity, \( C_{\omega_z} \).
**Characteristics of Convection**  Convection is characterized visually, as well as quantitatively via measurements of length scales, columnarity, helicity, flow speeds, heat transfer, and volume-integrated rms forces.

**Flow Visualizations**  Figure 9 shows snapshots of axial vorticity $\omega_z = (\nabla \times \mathbf{u}) \cdot \hat{z}$ isosurfaces for select dynamo and rotating convection models. These flow visualizations illustrate that the axial alignment of convective structures outside of the tangent cylinder persists from near onset to $Ra \approx 19Ra_c$, regardless of the presence of magnetic fields. (The tangent cylinder is the imaginary right cylinder tangent to the inner shell at the equator.) Above $Ra \approx 19Ra_c$, columnar convection is destroyed in both dynamo and non-magnetic models and the flows become strongly three-dimensional. This flow transition, which defines the boundary between Regimes II and III, is controlled by the relative thicknesses of the Ekman and thermal boundary layers and occurs at $Ra \approx 1.4Ra_t$.

**Length Scales**  Linear theory predicts there to be a fundamental change in characteristic length scale from $m = \mathcal{O}(E^{-1/3}D)$ as $E \to 0$ in non-magnetic systems (Roberts, 1968; Jones et al., 2000; Dormy et al., 2004) to $m = \mathcal{O}(D)$ when a strong magnetic field with $\Lambda_i \gtrsim \mathcal{O}(1)$ is imposed in the limit $E \to 0$ (Chandrasekhar, 1961; Eltayeb and Roberts, 1970; Fearn and Proctor, 1983; Cardin and Olson, 1995).
We quantify length scales using the characteristic wavenumber of flow, $k_u$:

$$k_u = \sqrt{l_u^2 + m_u^2},$$  \hspace{1cm} (28a)

where $l_u = \sum_{l=0}^{l_{max}} l (u_l \cdot u_l) / 2E_K$  \hspace{1cm} (28b)

and $m_u = \sum_{m=0}^{m_{max}} m (u_m \cdot u_m) / 2E_K$.  \hspace{1cm} (28c)

Here, $u_l$ is the velocity at spherical harmonic degree $l$, $u_m$ is the velocity at spherical harmonic order $m$, and $E_K$ is the kinetic energy. The time-averaged values, given in Table 6, show that the presence of dynamo-generated magnetic fields alters the value of $k_u$ by at most 15% in comparison to the associated non-magnetic cases. Although this change is significant, it is not the fundamental change in length scale that linear theory predicts.

**Columnarity**  We quantify the style of convection using axial vorticity measurements. Quasigeostrophic convection (Regimes I and II) is dominated by axial, vortical columns that extend in $\hat{z}$ across the shell. We define ‘columnarity’ using a measure of the axial variations of $\omega_z$ in the bulk fluid outside of the tangent cylinder:

$$C_{\omega z} = \frac{\sum_{s,\phi} |\langle \omega' \cdot \hat{z} \rangle_z|}{\sum_{s,\phi} |\langle |\omega'| \rangle_z|}. \hspace{1cm} (29)$$

Here, $\langle \rangle_z$ indicates averages in the axial $\hat{z}$ direction, $\omega'$ indicates vorticity calculated using only the non-axisymmetric velocity field, and the summation occurs over the equatorial plane $(s, \phi)$. Columnar convection has relatively large columnarity, $C_{\omega z} \gtrsim 0.5$, because vorticity, $\omega'$, is dominated by its axial component, $\omega' \cdot \hat{z}$. Thus,
we consider cases with $C_{\omega z} \gtrsim 0.5$ to be columnar, similar to our convention for $f$. Comparison of axial vorticity isosurfaces shows this convention to be an adequate proxy for the breakdown of columnar convection.

Figure 10a shows columnarity as a function of the Rayleigh number. The $C_{\omega z}$ values agree to within an average of 4% between the dynamo and non-magnetic models, with a maximum difference of 12%. The presence of magnetic fields, therefore, does not change the basic planform of convection.

The columnarity values decrease with increasing $Ra$. There is no first order transition in $C_{\omega z}$, as occurs in the dipolarity data. However, following our convention, we argue that the flow becomes dominated by three-dimensional motions when $C_{\omega z} < 0.5$. This occurs near $Ra = 19Ra_c$. This flow transition does not coincide with the magnetic field morphology transition at $Ra = 5.1Ra_c$. Therefore, columnar convection can generate both dipolar (Regime I) and multipolar (Regime II) magnetic fields. We find, however, that this breakdown occurs at $Ra/Ra_t = 1.4$, in agreement with the boundary layer arguments of King et al. (2009) and King et al. (2010).
Figure 10: Instantaneous a) axial vorticity columnarity, $C_{\omega z}$, and b) relative axial helicity, $|H_z^*|$, versus the Rayleigh number. In each plot, the solid vertical line indicates the sharp transition between dipolar and multipolar dynamos, and the long-dashed vertical line indicates where the convective flow structures are no longer axial.
Helicity  Convective flows can also be characterized by their helicity. Helicity describes corkscrew-like motions as measured by correlations between velocity and vorticity fields. Here, we consider axial helicity $H_z = u_z \omega_z$. Relative axial helicity is defined as axial helicity normalized by its maximum possible value,

$$\begin{align*}
H_{z,rel} &= \frac{\langle H_z \rangle_h}{\left(\langle u_z u_z \rangle_h \langle \omega_z \omega_z \rangle_h\right)^{1/2}},
\end{align*}$$

(30)

where $\langle \rangle_h$ is the volumetric average, excluding boundary layers, in each hemisphere (e.g., Olson et al., 1999; Schmitz and Tilgner, 2010). Since axial helicity tends to be anti-symmetric across the equator, we report the average helicity magnitude averaged over both hemispheres, $|H_{z,rel}|$.

Figure 10b shows calculations of relative axial helicity plotted versus the Rayleigh number. Helicity is not appreciably sensitive to the presence of magnetic fields; our dynamo and non-magnetic models produce $|H_{z,rel}|$ values that typically differ by less than 10%. Helicity is diminished by increased thermal forcing ($Ra$). Regime I models exhibit helical flows with $|H_{z,rel}| \gtrsim 0.4$. Near the Regime I–II boundary ($3.8 \leq Ra/Ra_c \leq 8.0$), helicity drops off significantly. The three-dimensional flows in Regime III models are poorly correlated, such that $|H_{z,rel}| \lesssim 0.1$. Thus, the columnar nature of the flow field does not significantly vary across the Regime I–II boundary, but the helicity in both dynamo and non-magnetic models do.
**Convective Flow Speeds** Convective flow speeds are given by the convective Reynolds number, \( Re_c = \frac{U_c D}{\nu} \), where \( U_c \) is the rms flow speed excluding the axisymmetric zonal flow component. Figure 11a plots the time-averaged convective Reynolds number versus the Rayleigh number. The non-magnetic models have \( Re_c \) values that are on average 14% stronger than those of associated dynamos, with a maximum difference of 24%. These differences likely exist because magnetic damping acts to reduce the flow velocities in the dynamos. Figure 11a shows that the convective flow speeds, however, are more sensitive to thermal driving (\( Ra \)) than the presence of magnetic fields.

The solid line in Figure 11a shows the classic \( Re \propto Ra^{1/2} \) ‘free-fall’ scaling law found in non-magnetic, non-rotating turbulent convection (e.g., Sano et al., 1989; Castaing et al., 1989; Siggia, 1994; Tilgner, 1996). Our data roughly follow this scaling in Regime III, which suggests that neither magnetic fields nor rotation strongly influence these cases.
Figure 11: Time-averaged a) convective flow speed, $Re_c$, and b) heat transfer efficiency, $Nu$, as a function of the Rayleigh number. The solid lines indicate classic scalings for non-rotating, non-magnetic convection: a) $Re_c \propto Ra^{1/2}$ and b) $Nu \propto Ra^{2/7}$. In each plot, the solid vertical line indicates the sharp transition between dipolar and multipolar dynamos, and the long-dashed vertical line indicates where the convective flow structures are no longer axial.
Heat Transfer  Figure 11b shows the time-averaged heat transfer behavior plotted versus the Rayleigh number. The Nusselt number, $Nu$, is the ratio of total to conductive heat transfer:

$$Nu = \frac{R_O}{R_I} \frac{qD}{\rho C_p \kappa \Delta T},$$

where $q$ is the heat flux per unit area on the outer shell boundary and $C_p$ is specific heat capacity. Between the dynamo and non-magnetic models, the Nusselt numbers agree to within an average of 3% with a maximum difference of 9%. In Regime I, the presence of magnetic fields produces slightly larger radial length scales (see Table 6) which can then transport heat more efficiently. In Regime III, magnetic fields weakly damp flow speeds (Figure 11a), tending to reduce heat transport. Despite these differences, the magnetic field has a second order influence on heat transport. The classic $Nu \propto Ra^{2/7}$ scaling law often found in non-magnetic, non-rotating turbulent convection systems (e.g., Castaing et al., 1989; Glazier et al., 1999) is superimposed in Figure 11b. A comparison of our data against this scaling further supports our contention that inertially-dominated convection occurs in Regime III.

Force Integrals  The momentum equation can be used to measure the competition between Lorentz ($F_L$), Coriolis ($F_C$), inertial ($F_I$), and viscous ($F_V$) forces. Towards this end, the dimensionless forces are integrated over the entire spherical shell volume:

$$F = \int_0^{2\pi} \int_0^\pi \int_{R_I}^{R_O} \left( F_r^2 + F_\theta^2 + F_\phi^2 \right)^{1/2} r^2 \sin \theta \, dr \, d\theta \, d\phi,$$

(32)
where $\mathcal{F}(r, \theta, \phi)$ is a generic force density given by each term in equation (19). Boundary layers are included in the integration, but their exclusion does not significantly affect the results, with the exception of the viscous force integral where up to 50% of the force is contained within the boundary layer.

Figures 12 and 13 shows these force integrals and their ratios for the dynamo and rotating convection models, respectively. In both sets of models, the Coriolis term dominates in Regimes I and II. These models are then in quasigeostrophic balance, which explains the prevalence of columnar convection within these regimes.

The Lorentz force is never a dominant influence on convection dynamics in our models. The ratio of Lorentz to Coriolis forces does not exceed 0.3 in Regimes I and II, while the ratio of Lorentz to inertial forces is less than 0.7 in Regimes II and III. This subdominance of the Lorentz force explains why our dynamo and rotating convection models exhibit similar convective (non-zonal) behaviors.
Figure 12: a) plots instantaneous integrals of the rms Coriolis, Lorentz, inertial, and viscous forces versus the Rayleigh number for the dynamo models. Ratios of the force integrals are shown in b). The inner (outer) color corresponds to the numerator (denominator) of the force ratio. In each plot, the solid vertical line indicates the sharp transition between dipolar and multipolar dynamos, and the long-dashed vertical line indicates where the convective flow structures are no longer axial.
Figure 13: a) plots instantaneous integrals of the rms Coriolis, Lorentz, inertial, and viscous forces versus the Rayleigh number for the non-magnetic models. Ratios of the force integrals are shown in b). The inner (outer) color corresponds to the numerator (denominator) of the force ratio. In each plot, the solid vertical line indicates the sharp transition between dipolar and multipolar dynamos, and the long-dashed vertical line indicates where the convective flow structures are no longer axial.
**Zonal Flow**  Figure 14 shows that a first-order change in the style of zonal flow also occurs between dipolar and multipolar dynamo models. Zonal flows exhibit significant axial variation in Regime I, but are axially-aligned outside the tangent cylinder in Regime II, similar to the non-magnetic models in both Regimes I and II.

---

**Figure 14:** Time-averaged zonal flows in **a**) Regime I dynamo model, **b**) Regime II dynamo model, **c**) Regime I non-magnetic model, **d**) Regime II non-magnetic model. Red (blue) indicates prograde (retrograde) flow. Each subplot has its own colorscale.
The zonal flow momentum is governed by the azimuthally-averaged $\hat{\phi}$ component of the momentum equation (19):

$$E \left( \frac{\partial u_\phi}{\partial t} + [u \cdot \nabla u]_\phi - [\nabla^2 u]_\phi \right) = -[u]_s + \left[ \frac{1}{2Pm} \nabla \times (\nabla \times B) \times B \right]_\phi,$$

(33)

where $u_\phi = [u]_\phi = [u] \cdot \hat{\phi}$ is zonal flow velocity, square brackets indicate the azimuthal average, and $s$ is cylindrical radius. The equilibrated zonal velocity field varies slowly in time such that first term on the left side of (33) is small. Further, there is no pressure gradient term because $[\nabla \Pi] \cdot \hat{\phi}$ is identically zero; thus, geostrophic balance cannot be established for the axisymmetric zonal component of flow. When the inertial and viscous forces are weak, the terms on the left side of (33) are small and the Coriolis force can only be balanced by the Lorentz force. The axisymmetric azimuthal forces are integrated over the meridional $(r, \theta)$ hemisphere, including the boundary layers. These force integrals and their ratios are plotted in Figure 15.

The zonal flow is in magnetostrophic balance in Regime I because the inertial and viscous terms are small. Once the dipole breaks down, the system is no longer magnetostrophic since the Lorentz force weakens and the zonal inertia increases in order to balance the Coriolis term. The zonal flows in multipolar dynamo models are then well-described by those in the corresponding non-magnetic models because the Lorentz force is too weak to significantly affect the dynamics.
Figure 15: a) Instantaneous integrals of the rms azimuthal axisymmetric Coriolis, Lorentz, inertial, and viscous forces versus the Rayleigh number. b) Ratios of the force integrals. In each plot, the solid vertical line indicates the sharp transition between dipolar and multipolar dynamos, and the long-dashed vertical line indicates where the convective flow structures are no longer axial.
3.4.3 Parameterization of Magnetic Field Influence

Below we compare calculations and parameterizations of the Lorentz to Coriolis force ratios in our models.

**Traditional Elsasser Number**  Following the development of Cardin et al. (2002), the ratio of the Lorentz force \((J \times B/\rho)\) to Coriolis force \((2\Omega \times u)\) is parametrized by the general form of the Elsasser number:

\[
\Lambda = \frac{JB}{2\rho\Omega U}.
\]  

(34)

In order to estimate \(\Lambda\) using rms magnetic field strength \(B\), the current density \(J\) is characterized either via Ohm’s law

\[
J = \sigma(E + u \times B)
\]

(35)

(where \(\sigma = 1/\mu_0\eta\) is the electrical conductivity) or via Ampere’s law under the MHD approximation

\[
J = \frac{1}{\mu_0} \nabla \times B.
\]

(36)

Using (35) in (34), the contribution to current density from the electrical field \(E\) is typically discarded, giving \(J \sim \sigma UB\) and yielding the traditional form of the Elsasser number:

\[
\Lambda_i = \frac{B^2}{2\rho\mu_0\eta\Omega}.
\]

(37)

Figure 16 compares \(\Lambda_i\) against the ratio of the Lorentz and Coriolis force integrals from Figure 12b. The explicitly calculated force integral ratios range over
0.02 \leq \frac{F_L}{F_C} \leq 1.6, with a mean of 0.3. This demonstrates that the magnetic field is dynamically weak with respect to the Coriolis force in most of our models. In contrast, the traditional Elsasser numbers are typically greater than unity, ranging from 0.2 \leq \Lambda_i \leq 200 with a mean value of 20. These large values of \Lambda_i incorrectly imply that Lorentz forces should dominate. The traditional form of the Elsasser number then overestimates the strength of the Lorentz force in the dynamo models, typically by a factor of approximately 10.

Figure 16: Comparison of the (instantaneous) calculated Lorentz to Coriolis force integral ratios against the (time-averaged) traditional and dynamic Elsasser numbers as a function of the Rayleigh number. The solid vertical line indicates the sharp transition between dipolar and multipolar dynamos, and the long-dashed vertical line indicates where the convective flow structures are no longer axial.
The misfit between \( \Lambda_i \) and the calculated force ratio can be understood in terms of the two main assumptions that are made to arrive at \( \Lambda = \Lambda_i \). First, this formulation assumes that \(|u \times B| = UB\), which is not necessarily appropriate in a non-linear system in which the flow and field can self-organize such that interaction is more limited: \(|u \times B| < UB\). Second, the assumption that \( E = 0 \) physically implies that the magnetic field is not strongly time-variant. This can be seen by combining (35) and (36) to obtain the uncurled magnetic induction equation:

\[
\sigma E + \sigma u \times B - \frac{1}{\mu_o} \nabla \times B = 0
\]  

(38)

The terms from left to right represent the time evolution, induction, and diffusion of magnetic field, respectively. By ignoring the contribution to current density from the electric field in (35) to get \( \Lambda_i \), temporal variations of the magnetic field are neglected. This assumption is likely valid for quasi-steady dynamo solutions and in MHD systems with imposed magnetic fields that do not vary strongly with time (\( Rm < 1 \)). However, most natural and simulated dynamos exhibit significant time variability (\( Re > 1, Rm > 1 \)). Therefore, the traditional Elsasser number, \( \Lambda_i \), may not accurately gauge the strength of the Lorentz force in dynamos.

**Dynamic Elsasser Number**  The strength of the Lorentz force can be estimated without making these assumptions by using the form of current density from Ampere’s law (36), so the current density can be parametrized as \( J \sim B/\mu_o \ell_B \) (e.g., Cardin et al., 2002). We characterize magnetic field gradients using a typical quarter-
wavelength of magnetic field variations: \( \ell_B \sim (\pi D/2)/\sqrt{k_B} \), where

\[
\bar{k}_B = \sqrt{l_B^2 + m_B^2},
\] (39)

analogous to \( \bar{k}_u \) (see Table 5). This allows us to define a ‘dynamic Elsasser number’, \( \Lambda_d \), in which the relative strength of Lorentz and Coriolis forces is estimated by

\[
\Lambda_d = \frac{B^2}{2\rho \mu_o \Omega U \ell_B} = \frac{\Lambda_i D}{Rm \ell_B}.
\] (40)

Figure 16 shows calculations of \( \Lambda_d \) from our dynamo models. The values range over \( 0.01 \leq \Lambda_d \leq 1.1 \) with a mean of 0.2, correctly predicting that the influence of magnetic fields on convection is secondary with respect to the Coriolis force in most of our models. Further, the dynamic Elsasser number is in good agreement with the Lorentz to Coriolis force integral ratios; the values differ by at most a factor of two.

The dynamic Elsasser number naturally arises when the momentum equation is scaled with dynamically relevant, local length and time scales. Thus, we non-dimensionalize using the rotational timescale, \( \tau_\Omega = 1/\Omega \), because the Coriolis force is first order in all of our models (see Figure 12). We further scale velocity as \( U \), velocity gradients as \( \nabla u \sim U/\ell_u \), velocity laplacian as \( \nabla^2 u \sim U/\ell_u^2 \), magnetic field intensity as \( B \), magnetic field gradients as \( \nabla B \sim B/\ell_B \), pressure as \( \rho \Omega U \ell_u \), and temperature as \( \Delta T \).

The dimensionless momentum equation is then:

\[
\frac{1}{2} \frac{\partial \mathbf{u}}{\partial t} + R_{\theta \ell} \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{z} \times \mathbf{u} = -\frac{1}{2} \nabla p + \frac{Ra^*}{R_{\theta \ell} r_o} T + E_\ell \nabla^2 \mathbf{u} + \Lambda_d (\nabla \times \mathbf{B}) \times \mathbf{B}.
\] (41)

Here, \( \ell_u \) is the length scale over which flow variations occur, \( \ell_B \) is the length scale
over which magnetic field variations occur, and $\ell_\nu$ is the length scale at which viscous diffusion occurs.

The new non-dimensional parameters are the local Ekman number $E_\ell$, which characterizes the ratio of viscous to Coriolis forces, the local Rossby number $Ro_\ell$, which characterizes the ratio of inertial to Coriolis forces, and the local modified Rayleigh number $Ra^*_{\ell}$, which characterizes the squared ratio of buoyancy to Coriolis forces:

$$E_\ell = \frac{\nu}{2\Omega \ell_\nu^2}, \quad Ro_\ell = \frac{U}{2\Omega \ell_u}, \quad Ra^*_{\ell} = \frac{\alpha g \Delta T}{4\Omega^2 \ell_u}. \quad (42)$$

In addition, the ratio of Lorentz to Coriolis forces in (41) is given by

$$\frac{F_L}{F_C} = \Lambda_d, \quad (43)$$

providing an additional explanation of why the dynamic Elsasser number agrees well with the calculated Lorentz to Coriolis force ratios in Figure 16.

**Lehnert Number** Another parameter used to characterize the competing roles of Lorentz and Coriolis forces in planetary dynamo models is the Lehnert number, $\lambda$ (Lehnert, 1954; Fearn et al., 1988). This parameter has been employed by recent studies considering the effects of imposed magnetic fields on transient motions in rapidly rotating spherical shells (Jault, 2008; Gillet et al., 2011). The Lehnert number quantifies this force ratio by comparing the rate of rotation to an Alfvén wave frequency. As such, $\lambda$ represents a special case of the dynamic Elsasser number, $\Lambda_d$, where the typical flow speed $U$ is assumed to scale as the Alfvén wave speed,
$V_A = B / \sqrt{\rho \mu_0}$. Substituting $U = V_A$ into (40) produces the Lehnert number:

$$\lambda = \frac{B}{2 \ell_B \Omega \sqrt{\rho \mu_0}} = A \Lambda_d,$$  \hspace{1cm} (44)

where the Alfvén number, $A = U / V_A$, is the ratio of the flow velocity to the Alfvén wave speed. Thus, when flow speeds follow the Alfvén wave speed scaling, $A = O(1)$, the Lehnert number should aptly characterize this force balance. However, in many planetary dynamo models, the flow speeds and magnetic field strengths are not found to be related by $A \approx 1$ (cf. Christensen and Aubert, 2006). Therefore, we argue that $\Lambda_d$ provides a more accurate estimation of the Lorentz to Coriolis force ratio.

**Applications to Dynamo Models** We have shown that the Lorentz to Coriolis force ratio is well-described by the dynamic Elsasser number in our models. However, our survey is carried out at a fixed, moderate Ekman number ($E = 10^{-4}$). In order to infer how $\Lambda_d$ trends with $E$, we can make use of the fact that the characteristic wavenumbers $k_u$ and $k_B$ are comparable for the moderate Prandtl number values ($Pr \approx Pm \approx 1$) used in most present day dynamo simulations (e.g., King et al., 2010). We, therefore, approximate the length scales of both magnetic field variations and convective flow structures using the characteristic azimuthal quarter-wavelength of the flow: $\ell \sim (\pi D / 2) / m_u$. Because the azimuthal wavenumber is found to scale as $m_u \sim E^{-1/3}$ in rapidly rotating convection models (e.g., Stellmach and Hansen, 2004; Gillet and Jones, 2006), we can express (40) as

$$\Lambda_d \sim \frac{\Lambda_i}{E^{1/3} Rm}.$$  \hspace{1cm} (45)
This implies that the traditional Elsasser number, $\Lambda_i$, and the dynamic Elsasser number, $\Lambda_d$, will become comparable when $E \sim Rm^{-3}$. Taking $Rm \sim 100$ and $\Lambda_i \sim 1$, we then predict that dynamo models will operate in local magnetostrophic balance when $E \lesssim 10^{-6}$. This prediction can be tested by current state-of-the-science simulations (e.g., Sakuraba and Roberts, 2009).

**Applications to Planetary Cores**  Application of the dynamic Elsasser number to planetary cores requires an estimate of the length scale over which magnetic variations occur. Metallic planetary core fluids are thought to have small magnetic Prandtl numbers ($Pm \lesssim 10^{-5}$; e.g., Dobson et al., 2000). Length scales for magnetic field variations should then be significantly larger than length scales of the velocity field. As a first order estimate, we assume that the magnetic field is large-scale ($\ell_B \sim D$). Then the dynamic Elsasser number can be written as

$$
\Lambda_d = \frac{B^2}{2\mu_0 \rho \Omega UD} = \frac{\Lambda_i}{Rm}. \tag{46}
$$

Using estimates for planetary dynamo regions of $\Lambda_i \lesssim 1$ and $Rm \gtrsim 100$ (Schubert and Soderlund, 2011), all planets with active magnetic fields are predicted to have $\Lambda_d$ values less than unity. Extrapolating our results to planetary settings, we predict that the non-zonal motions in planetary interiors are not strongly influenced by their large-scale magnetic fields. This is consistent with analysis of geomagnetic secular variation data, which suggests that large-scale flows in Earth’s core are quasi-geostrophic and, therefore, are not strongly influenced by magnetic fields (Schaeffer
and Pais, 2011).

### 3.4.4 Breakdown of Dipolar Magnetic Field Generation

We observe a transition from dipolar to multipolar magnetic fields at the Regime I–II boundary (Figure 8). Poloidal magnetic fields, including the dipole component, are generated by the $\alpha$-effect in planetary dynamo models (Christensen and Wicht, 2007; Wicht and Tilgner, 2010). The $\alpha$-effect describes the generation of large-scale fields by strongly correlated smaller-scale flow, which is typically attributed to the helical nature of rotating convection (e.g., Parker, 1955).

In Figure 10b, we observe that helical flow also breaks down at the Regime I–II boundary. This suggests that the degradation of helices may be responsible for the breakdown of dipolar field generation. Furthermore, the change in helicity across this regime boundary also occurs in the absence of magnetic fields, which indicates that the mechanism responsible for the transition is purely hydrodynamic. The mechanism itself, however, is not well understood.

Several studies have suggested that the breakdown of dipolar field generation is the result of a competition between inertial and Coriolis forces such that $Ro_i \gtrsim 0.1$ (e.g., Sreenivasan and Jones, 2006; Christensen and Aubert, 2006; Olson and Christensen, 2006). We observe in Figure 12b, however, that the calculated Coriolis force is an order of magnitude stronger than inertia where this field morphology transition occurs.
The dominance of the Coriolis force in both regimes suggests that the transition in helical flow and magnetic field morphology is caused instead by competition between second-order forces. Specifically, at the Regime I–II boundary, we find that inertia becomes stronger than viscosity.

We hypothesize that, in the vast majority of present day dynamo models (e.g., $E \gtrsim 10^{-4}$ and $Pm \sim Pr \sim 1$), viscosity plays an important role in the stability of helical flow, and, therefore, in the generation of dipolar fields. If the helices that generate large-scale magnetic fields in dynamo models are viscously-controlled, however, it is unlikely that these models correctly reproduce the physical mechanisms of field generation in planetary cores where viscosity is negligible and Ohmic dissipation dominates viscous dissipation (Verhoogen, 1980). Future work on the stability of helical flow is clearly needed.

### 3.4.5 Conclusions

We have carried out a broad survey of dynamo and non-magnetic rotating convection models in which the array of behaviors are mapped as a function of thermal forcing. Three dynamo regimes are identified: I) columnar convection with dipole-dominated dynamos, II) columnar convection with multipolar dynamos, and III) three-dimensional convection with multipolar dynamos.

Comparisons of dynamos against otherwise identical non-magnetic models indicate
that the characteristics of convection (axial vorticity isosurfaces, characteristic length scales, columnarity, relative axial helicity, convective flow speeds, heat transfer efficiency, and volume-integrated rms forces) are not significantly affected by magnetic fields. We show that the traditional Elsasser number definition, $\Lambda_i$, is not an appropriate measure of the relative strengths of Lorentz and Coriolis forces in dynamo systems. Instead, we argue that the dynamic Elsasser number, $\Lambda_d$, better parametrizes this ratio of forces. Extrapolating our results to planetary cores, we predict that the Lorentz force due to observable, large-scale magnetic fields are at least 100 times weaker than the Coriolis force.

Our results also suggest that the collapse of dipolar magnetic fields is due to the breakdown of helical flow. The prevailing idea in the literature is that the breakdown of helical flow is controlled by the ratio of inertial and Coriolis forces (e.g., Olson and Christensen, 2006). This explanation, however, lacks a clear physical mechanism responsible for the transition. Instead, we demonstrate that helical flow structures breakdown once the destabilizing inertial effects overcome the stabilizing influence of the viscous diffusion force. Large-scale field generation in present day terrestrial-style dynamo models, therefore, appears to be accomplished by viscously stabilized helices.
4 Models of Ice Giant-style Dynamos

4.1 Introduction

Three magnetic field morphologies are observed in the solar system (Table 1). Earth, Ganymede, and the gas giants have dipole-dominated magnetic fields that are nearly aligned with their rotation axes, while Mercury is dominated by axially-aligned dipolar and quadrupolar field components. In contrast, the ice giants have non-axisymmetric multipolar magnetic fields with quadrupole and octupole components that are comparable to or greater than the dipole. This chapter focuses on such multipolar dynamos.

Other observations of the giant planets also sharply differ between the gas and ice giants, particularly the atmospheric winds and thermal emissions. The zonal flows, thermal emissions, and magnetic fields of Uranus, Neptune, and Jupiter are shown in Figure 17 to contrast the behaviors. Despite these fundamental differences, little work has been done to examine the dichotomy between the ice giants and the gas giants.

For instance, the leading model of ice giant-style dynamos is that of Stanley and Bloxham (2004, 2006). Their model rests upon the idea that dynamo action in Uranus and Neptune occurs in a geometrically thin layer that must lie above a convectively stable ionic ocean. New internal structure models, however, find that a thin convecting layer
may exist on Uranus, but not on Neptune (Fortney et al., 2011). Thus, the model of Stanley and Bloxham (2004, 2006) cannot explain the dynamo observations of both bodies. Furthermore, there is no clear mechanism given in their work to explain why an underlying stably-stratified ocean leads to multipolar dynamos. Because of the significant agreement between Uranus’ and Neptune’s magnetic fields, zonal flows, and thermal emission fields, I argue that an alternative model is needed to provide a unified explanation of the Ice Giant observations.

Towards this end, I have carried out a four-case study to investigate ice giant-style dynamics. We test the hypothesis that gas and ice giants differ due to the relative influence of inertial to Coriolis effects by carrying out strongly-forced planetary dynamo models and comparing them against observations of Uranus and Neptune. Our models show that inertially-dominated convection generates multipolar magnetic fields, zonal flows, and thermal emission patterns that qualitatively agree with the observations, suggesting that this convective regime may be relevant to the ice giants.

### 4.2 Ice Giant Observations

There are three global observations of the ice giants, Uranus and Neptune, that hint toward their internal dynamics: magnetic fields, atmospheric winds, and thermal emissions. Uranus and Neptune are the only known bodies to have non-axisymmetric, multipolar magnetic fields, where the quadrupole and octupole components are com-
parable to or greater than the dipole. Higher order components are presently below the limits of spatial resolution, and there is no information about secular variation.

The surface zonal winds are determined by tracking small cloud features over short time intervals. Observations made by the Voyager 2 spacecraft, the Hubble Space Telescope, and ground-based telescopes revealed that the surface winds are dominated by zonal (east-west) motions (e.g. Smith et al., 1986, 1989; Sromovsky et al., 2001; Hammel et al., 2001, 2005; Sromovsky et al., 2009; Karkoschka, 2011). These zonal winds are measured with respect to the mean rotational motion of the planet, assumed to correspond with the deep-seated magnetic field frame of reference (e.g., Ness et al., 1994). Voyager 2 radio observations suggest that the rotation periods of Uranus and Neptune are 17.24 hours (Desch et al., 1986; Warwick et al., 1986) and 16.11 hours (Desch et al., 1986; Warwick et al., 1989), respectively. Adopting these periods, both planets have retrograde (westward) jets near the equator and prograde (eastward) jets at high-latitudes, as shown in Figure 17. Further, Neptune has the fastest wind speeds in the solar system that approach −400 m/s in the equatorial jet and +250 m/s in the high latitude jets. Uranian winds have maximum velocities near +200 m/s in the high latitude jets and an equatorial jet speed of about −100 m/s.
Figure 17: Surface observations of zonal flow, heat flux, and radial magnetic fields. The dimensionless heat flux profiles have been normalized such that the maximum value is unity. Zonal flow data is taken from Sukoriansky et al. (2002) (short-dashed line) and Helled et al. (2010) (long-dashed line) for the ice giants and Porco et al. (2003) for Jupiter, thermal emission data is from Pearl et al. (1990), Pearl and Conrath (1991), and Pirraglia (1984), and magnetic field data is from Holme and Bloxham (1996) and Yu et al. (2010).
There are complications, however, in determining the rotation rates from radio observations. For example, the radio emissions of Saturn have multiple periodicities and vary in time (Gurnett et al., 2007, 2009). Alternatively, dynamical arguments can be used to estimate the rotation periods (Helld et al., 2010; Karkoschka, 2011). Helled et al. (2010) estimate the rotation periods by minimizing the wind velocities and find periods that differ from those based on Voyager 2 radio signals by -40 minutes and +68 minutes for Uranus and Neptune, respectively. In contrast, Karkoschka (2011) estimates the rotation period of Neptune to be 15.97 hours, a difference of -8 minutes from the Voyager 2 derived period, by tracking features (e.g., storms) and noting their rotational stabilities. These differences illustrate the uncertainties in giant planet rotation rates, which is reflected in the zonal flows. Adopting the Helled et al. (2010) rotation periods, both Uranus and Neptune have retrograde equatorial jets with speeds of about -150 m/s and prograde high latitude jets with peak speeds of about +200 m/s. Zonal flow profiles assuming the Helled et al. (2010) rotation periods are superimposed in Figure 17 to illustrate these differences.

The Voyager 2 flybys also measured the thermal energy emitted by Uranus and Neptune. Heat flux as a function of latitude is also given in Figure 17. The heat flux pattern of Neptune peaks in the equatorial and polar regions with minima at mid-latitudes (Pearl and Conrath, 1991); the heat flux pattern of Uranus is not well constrained (Pearl et al., 1990). Interestingly, the energy balances between Uranus and Neptune sharply contrast. Neptune emits more than twice the thermal energy
it receives via insolation, while the total emission to insolation ratio is no greater than \( \sim 1.1 \) for Uranus (Pearl and Conrath, 1991; Pearl et al., 1990). These ratios imply internal heat sources due to the release of primordial heat of formation and/or radioactive heating. Uranus and Neptune have internal heat fluxes of 0.04 W/m\(^2\) and 0.4 W/m\(^2\), respectively (Pearl and Conrath, 1991; Pearl et al., 1990). Convection is thought to occur in the interiors of both Uranus and Neptune since this internal heat flux exceeds the adiabatic heat flux estimates of 0.01 W/m\(^2\) (e.g., Aurnou et al., 2007; Fortney et al., 2011) and since they have internally generated dynamos. However, a layer stable to convection in Uranus’ interior due to significant compositional gradients has been proposed to explain the energy balance difference (Stevenson, 1987; Podolak et al., 1991, Fortney et al., 2011; cf. Holme and Ingersoll, 1994).

The generation of the ice giants’ winds and magnetic fields are intrinsically linked to their internal structures and dynamics. The interiors, which are poorly constrained, are often modeled as three nested regions: \( i \) an outermost molecular envelope largely composed of hydrogen and helium, \( ii \) a weakly conducting fluid mixture of \( \text{H}_2\text{O}, \text{CH}_4, \) and \( \text{NH}_3 \) known as the “ice layer”, and \( iii \) a central core (e.g. Hubbard et al., 1991; Podolak et al., 1991; Guillot, 1999b). Note that no discontinuities between such regions is required (Helled et al., 2011). Schematics of possible internal structures of Uranus and Neptune are illustrated in Figure 18a, b.
Figure 18: Possible internal structure models of a) Neptune and b) Uranus inferred from Hubbard et al. (1991); Podolak et al. (1991); Lee et al. (2006); and Fortney et al. (2011). $R_P$ is planetary radius. Geometry of the c) Neptune-like and d) Uranus-like numerical models. $R_O$ is the outer shell radius of the model and is assumed to correspond to the planet surfaces in our simplified models that neglect electrical conductivity (and density) stratification.
Efforts toward understanding zonal wind generation in the giant planets primarily consist of two types: atmospheric climate models and deep convection models (e.g., Vasavada and Showman, 2005; Stanley and Glatzmaier, 2010). These models differ in the energy sources that drive the zonal motions and offer fundamentally different explanations for the observed zonal flows (Showman et al., 2006). Climate models assume the zonal flows to be driven in a shallow tropospheric layer (Cho and Polvani, 1996; Lian and Showman, 2010; Liu and Schneider, 2010), while deep convection models assume that the zonal flows are driven by convective motions in the planetary interior (e.g., Busse, 1976; Suomi et al., 1991; Heimpel et al., 2005; Aurnou et al., 2007).

Wind observations provide evidence towards the latter. Karkoschka (2011) argues that the stationary features found in Hubble Space Telescope images of Neptune may result from deep convective processes. Furthermore, the dominant motions observed on Uranus are perpendicular to the rotation axis, despite a dramatically different pattern of solar heating due to the planet’s 98° inclination with respect to the ecliptic. Instead of insolation peaking near the equator year round as for the other planets, Uranus’ polar regions are oriented towards the Sun during the summer and winter seasons. This has implications for what drives the zonal winds, insolation versus deep convection, and suggests that the winds may have a deep origin. We, therefore, choose to focus on deep convection models.

The magnetic fields of the ice giants are thought to result from convectively-driven
dynamo action in the ice layers (Ruzmaïkin and Starchenko, 1991; Stanley and Bloxham, 2004; Stevenson, 2010). Shock experiments on precompressed water suggest that the electrical conductivities in the ice giants are large enough to support dynamos at depths below about 0.8\(R_p\) (Cavazzoni et al., 1999; Lee et al., 2006). A relatively shallow dynamo region is consistent with the prominence of the higher order spherical harmonics of the planets’ magnetic fields (Russell and Dougherty, 2010).

The dynamics of the ice giants are fundamentally different from those observed on the gas giants, Jupiter and Saturn. The zonal flows, thermal emissions, and magnetic fields of Jupiter are shown in Figure 17 to show these differences. Jupiter’s surface winds are organized into a strong prograde equatorial jet and multiple jets at high latitudes that alternate in direction (Porco et al., 2003). In addition, the heat flux peaks near the poles with minimal flux in the equatorial region (Pirraglia, 1984). Further, the magnetic field is dipole-dominated with the dipole tilted about 10° from the rotation axis. Efforts toward understanding the dynamics of Uranus and Neptune must then also explain why these bodies are different from Jupiter and Saturn. We argue that this dichotomy is best explained by a difference in interior dynamics.

Since there are likely no sharp structural boundaries in the gas planets between the dynamo region and the overlying electrically insulating layer (cf. the core-mantle boundary of Earth), it is possible that the zonal flow, heat flow, and dynamo are all dynamically coupled together. Thus, we should be striving to generate models that can unify all the gas planet geophysical observations.
The hypothesis that the gas and ice giants differ due to the relative influence of inertial to Coriolis effects is motivated in Section 4.3. Section 4.4 details the models used to test whether strongly-forced dynamo models are characterized by ice giant-style dynamical patterns. Sections 4.5 – 4.7 investigate flow characteristics and generation, convective heat transfer, and magnetic field generation, respectively. Our results are compared against observations in Section 4.8. Conclusions are given in Section 4.9.

4.3 Dynamical Regimes

Previous work has shown that deep convection in rapidly-rotating spherical shells can drive zonal jets that are qualitatively similar to those observed on the gas giants (Heimpel et al., 2005; Aurnou et al., 2008; Jones and Kuzanyan, 2009), while inertially-dominated convection can drive zonal jets that are qualitatively similar to those observed on the ice giants (Gilman, 1977; Aurnou et al., 2007; Brun and Palacios, 2009). These dynamical regimes are discussed below.

4.3.1 Gas Giant-style Dynamics

Busse (1976) first proposed that rapidly-rotating deep convection in the molecular envelope may be responsible for the zonal winds of Jupiter. He envisioned that the alternating jets are the surface expressions of differentially-rotating cylinders aligned
with the rotation axis. In this hypothesis, the fluid motions are quasigeostrophic and do not strongly vary along the direction of the rotation axis, in agreement with the Taylor-Proudman theorem (Taylor, 1917; Proudman, 1916). Gas giant-like zonal flows with multiple jets were first obtained in rapidly-rotating Boussinesq convection models by Christensen (2001, 2002). More recently, Heimpel et al. (2005), Heimpel and Aurnou (2007), and Aurnou et al. (2008) have shown that turbulent, rapidly-rotating convection in thin spherical shells produces zonal flow and heat flux patterns that are consistent with those observed on Jupiter. The simulated zonal winds of the Heimpel et al. (2005) model are shown in Figure 19.

Figure 19: Instantaneous zonal winds on the inner and outer boundaries of the Heimpel et al. (2005) model. Red (blue) indicates prograde (retrograde) flow.

The equatorial jets in these models result from Reynolds stresses due to correlations between the fluctuating cylindrical and azimuthal velocity components (e.g., Zhang, 1992). Outside of the tangent cylinder, the imaginary right cylinder tangent to the inner shell at the equator, the height of the convection columns decreases with in-
creasing cylindrical radius. In order to conserve potential vorticity, $PV = (\omega_z + f)/h$

where $\omega$ is local fluid axial vorticity, $f = 2\Omega$ is planetary vorticity, and $h$ is axial fluid height, the fluid vorticity will readjust such that fluid moving away from the rotation axis will speed up relative to the inertial frame and fluid moving toward the rotation axis will slow down. This leads to spiraling convection cells in which cylindrically outward flows correlate with prograde azimuthal flows, and vice versa. These Reynolds stresses transport positive angular momentum outward and negative angular momentum inward, driving zonal flows outside of the tangent cylinder that are prograde near the outer boundary and retrograde near the inner boundary. Within the tangent cylinder of these models, quasigeostrophic turbulence leads to multiple jets whose widths are adequately determined by the Rhines scale (Rhines, 1975; Heimpel and Aurnou, 2007).

This style of zonal flow inhibits equatorial heat transfer and promotes polar heat transfer (Aurnou et al., 2008). In the equatorial plane, the convective flows are less efficient at transferring heat outward since they must compete with the strong azimuthal zonal flows. Convective motions in the polar regions are most efficient because zonal flows do not inhibit radial motions. Thus, quasigeostrophic deep convection can simultaneously explain the basic pattern of zonal jets and thermal emission on the gas giants.

Rapidly-rotating anelastic models also generate outer boundary zonal winds characterized by prograde equatorial jets and multiple smaller-scale jets at higher latitudes.
that alternate in direction. (Jones and Kuzanyan, 2009; Kaspi et al., 2009; Stanley and Glatzmaier, 2010). Jones and Kuzanyan (2009) and Kaspi et al. (2009) contrast anelastic and Boussinesq convection in rotating spherical shell models and show that the outer boundary zonal flow pattern is not strongly affected by the density stratification. The zonal flows differ at depth, however, where little vertical shear develops in the Jones and Kuzanyan (2009) models but significant vertical shear develops in the anelastic Kaspi et al. (2009) models; the reason for this difference has not yet been determined. Kaspi et al. (2009) also show that heat is preferentially emitted at high latitudes in their gas giant-like anelastic models, consistent with planetary observations and with the rapidly-rotating Boussinesq models. Further, Gary Glatzmaier’s anelastic gas giant-like dynamo simulation with radially-varying electrical conductivity, presented in Stanley and Glatzmaier (2010), shows that dynamo action in this rapidly-rotating convective regime simultaneously produces zonal flows and radial magnetic fields that agree to first order with the observations of Jupiter and Saturn. These Boussinesq and anelastic models, therefore, support the hypothesis that rotationally-dominated convection is relevant to the gas giants.

4.3.2 Ice Giant-style Dynamics

The models of Stanley and Bloxham (2004, 2006) are most cited to explain the magnetic fields of Uranus and Neptune. These models produce multipolar dynamos when moderately strong convection occurs in a thin spherical shell overlying a stably-
stratified fluid. They suggest that solid, electrically conducting inner cores act to stabilize the magnetic field and promote dipolar dynamos, while stably-stratified fluid cores do not have this anchoring effect. However, no mechanism explaining the generation of multipolar dynamos is given. While this model may be appropriate for Uranus, it is inconsistent with internal structure models of Neptune which do not predict a stably-stratified layer (e.g., Fortney et al., 2011). Thus, this hypothesis appears to be inconsistent with Neptunian internal structure models. An alternative model is then needed to provide a general explanation of ice giant-style dynamics.

Many dynamo models can generate multipolar fields (e.g., Grote et al., 1999, 2000a; Christensen et al., 1999; Kutzner and Christensen, 2000, 2002; Simitev and Busse, 2005; Sreenivasan and Jones, 2006; Gomez-Perez and Heimpel, 2007), and if geometrical reasons do not make Uranus and Neptune unique, it is not yet clear which type of model best explains the ice giants. The ability of rapidly-rotating deep convection models to explain both the zonal flow and heat flux patterns of the gas giants suggests that it may be difficult to use such models to explain the very different patterns observed on the ice giants. Consequently, differences in the convective flows may be responsible for gas giant-style versus ice giant-style models. Based on the success of the rotationally-dominated, gas giant-like models to explain the zonal flows, thermal emission patterns, and large-scale magnetic fields of Jupiter and Saturn (e.g., Heimpel et al., 2005; Aurnou et al., 2008; Stanley and Glatzmaier, 2010), we hypothesize that
inertially-dominated dynamo models will generate ice giant-style dynamics. As such, results of inertially-dominated convection modeling efforts are discussed below.

The non-magnetic Boussinesq models of Aurnou et al. (2007) and the non-magnetic anelastic models of Brun and Palacios (2009) and Kaspi et al. (2009) show that inertial convection in rotating spherical shells can drive zonal flows that qualitatively agree with the observations of Uranus and Neptune. In these models, the Taylor-Proudman constraint is relaxed by strong inertial effects, leading to three-dimensional convection. This convective turbulence strongly mixes the fluid and homogenizes the temperature and absolute angular momentum (e.g., Scorer, 1966; Gough and Lynden-Bell, 1968; Bretherton and Turner, 1968). When this mixing occurs, Aurnou et al. (2007) find that jets similar in morphology to those observed on the ice giants develop in order to conserve angular momentum.

Dynamo models driven by an ice giant-style zonal flow imposed on the outer boundary have been shown to generate magnetic fields where the axisymmetric quadrupolar component is dominant (Guervilly et al., 2011). Additionally, multipolar dynamos are typically coupled with substantial inertial effects (e.g., Grote et al., 1999, 2000a; Kutzner and Christensen, 2000, 2002; Simitev and Busse, 2005; Sreenivasan and Jones, 2006; Christensen and Aubert, 2006). This coupling is consistent with our hypothesis.
4.3.3 Convective Regime Transition

The boundary between rotationally-dominated and inertially-dominated convection is traditionally thought to be determined by the convective Rossby number, $Ro_c$. This parameter represents the ratio of inertial to Coriolis forces when the flow speed scales as the free-fall velocity. The free-fall velocity of convective structures can be obtained by assuming a balance between the inertial and buoyancy forces:

$$(u \cdot \nabla)u = \alpha_o T' g$$

This expression scales as $U^2/D \approx \alpha \Delta T g$, which leads to $U \approx \sqrt{\alpha \Delta T g D}$. Using this free-fall velocity in the Rossby number gives the convective Rossby number,

$$Ro_c = \left(\frac{\alpha g \Delta T}{4 \Omega^2 D}\right)^{1/2} = \sqrt{\frac{Ra E^2}{Pr}}.$$ 

This force balance argument predicts the convective regime transition to occur at $Ro_c \sim 1$ (e.g., Gilman, 1977, 1978; Aurnou et al., 2007; Zhong and Ahlers, 2010). However, if $Ro_c$ does correctly predict this transition, it will be difficult to reach $Ro_c$ values near unity since the rotation timescales are far shorter than the free-fall timescales.

Recent studies, however, have shown that the convective Rossby number does not accurately predict the parameter values where the velocity and heat transfer behaviors characteristically change (Sprague et al., 2006; King et al., 2009; Schmitz and Tilgner, 2009, 2010; Niemela et al., 2010). Instead, the transition between geostrophic and
inertial convection occurs at \( Ro_c < 1 \). As shown in Figure 20, a transition between columnar and three-dimensional flow clearly occurs in both laboratory and numerical experiments, despite the Rossby number being less than unity in all cases. This disparity suggests that it is easier to reach the inertially-dominated convective regime than previously thought. The mechanism responsible for the breakdown of columnar convection, however, is still debated.

Figure 20: a) and b) Shear structures in a cylindrical geometry laboratory experiment by King and Aurnou (2011); rotation is in the vertical direction. c) and d) Temperature fluctuations in planar geometry numerical simulations of Sprague et al. (2006) and Grooms et al. (2010). Despite all laboratory and numerical experiments shown here having \( Ro_c < 1 \), the convection is columnar in panels a) and c) but three-dimensional in panels b) and d). This result indicates that the convective Rossby number does not control the convective regime transition. Figure provided by J.M. Aurnou.
The most precise convective regime transition scalings to date are from King et al. (2009, 2010). These studies suggest that convective regimes may be controlled by boundary layer physics, particularly by the relative thicknesses of the thermal and Ekman boundary layers. With no other precise transition scalings on hand, we apply the King et al. (2010) scaling arguments, discussed in Chapter 3 and recast here in terms of the flux Rayleigh number, to planetary settings in order to make very rough estimates of the convective regimes in their interiors. The flux Rayleigh number is beneficial because it is defined using the heat flux, $Q$, rather than the superadiabatic temperature contrast, $\Delta T$, which is difficult to measure in nature:

$$\textit{Ra}_f = \frac{\alpha g_o Q D^4}{\rho c_p \kappa^2 \nu} = \textit{Ra} \textit{Nu}. \quad (49)$$

King et al. (2010) predict that the transition between rapidly-rotating and inertially-dominated convective regimes scales with the flux Rayleigh number, $\textit{Ra}_{f,t}$, as

$$\textit{Ra}_{f,t} = \textit{Ra}_t \textit{Nu}_t \sim E^{-9/4}. \quad (50)$$

The dynamics are predicted to be rotationally-dominated when $\textit{Ra}_f / \textit{Ra}_{f,t} < 1$ and inertially-dominated when $\textit{Ra}_f / \textit{Ra}_{f,t} \geq 1$. By comparing estimates of the planetary flux Rayleigh numbers to the transitional values, we can attempt to predict the style of convection in the planets’ interiors.

Table 8 compares estimates of the flux Rayleigh numbers of the planets’ dynamo regions against the transitional values predicted by (50). While the planetary flux Rayleigh numbers and the transition scaling are not yet known with confidence, our
estimates suggest that the dynamo regions of Uranus and Neptune are the most super-
critical with respect to the transition between rotationally-dominated and inertially-
dominated convection regimes. These arguments, therefore, suggest that an inertial
model of dynamo action on these bodies is worthy of investigation.

<table>
<thead>
<tr>
<th>Dynamo Region</th>
<th>$Ra_f$</th>
<th>$Ra_{f,t}$</th>
<th>$Ra_f/Ra_{f,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jupiter</td>
<td>$10^{41}$</td>
<td>$10^{43}$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>Saturn</td>
<td>$10^{37}$</td>
<td>$10^{41}$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Uranus</td>
<td>$10^{36}$</td>
<td>$10^{36}$</td>
<td>1</td>
</tr>
<tr>
<td>Neptune</td>
<td>$10^{39}$</td>
<td>$10^{36}$</td>
<td>$10^{3}$</td>
</tr>
</tbody>
</table>

Table 8: Order of magnitude estimates of the flux Rayleigh numbers for the planets’ dynamo regions compared against the transition flux Rayleigh number predicted by predicted by (50). Heat flux estimates are taken from Hanel et al. (1981) for Jupiter, Hanel et al. (1983) for Saturn, Pearl et al. (1990) for Uranus, and Pearl and Conrath (1991) for Neptune. The thermal expansion coefficients are taken to be $\alpha \sim 10^{-4}$ K$^{-1}$ for gas giants (e.g., Mills et al., 1977) and $\alpha \sim 10^{-3}$ K$^{-1}$ for ice giants (e.g., Croft et al., 1988). The specific heat capacity is taken to be $C_p \sim 10^3$ J / kg K for all bodies (e.g., Mills et al., 1977; Stacey, 2007). The other properties are taken from Tables 3 and 4.
4.4 Numerical Model

Few dynamo models in the inertially-dominated dynamical regime exist in the literature. Towards better understanding this regime, I have carried out four strongly-forced simulations to test the hypothesis that inertially-dominated convection in planetary dynamo models will produce zonal flow, heat flux, and magnetic field patterns that are consistent with observations of the ice giants.

Figure 18c, d shows the model setup. Two spherical shell geometries are considered in order to investigate the influence of convective-region geometry: a thick, Neptune-like shell with $\chi = 0.35$ and a thin, Uranus-like shell with $\chi = 0.75$. The fluid shells have stress-free, isothermal boundaries, and the inner spheres have the same electrical conductivities as the convecting outer shells. Stress-free boundary conditions are employed in order to minimize viscous diffusion (cf. Kuang and Bloxham, 1997; Roberts and Aurnou, 2011).

For simplicity, Boussinesq fluids are assumed where radial variations in electrical conductivity and density are neglected. This assumption is not physically realistic for ice giants, however, since it neglects the gradient in electrical conductivity between the nearly insulating molecular envelopes and the conducting ionic oceans. Furthermore, the convecting regions of the ice giants are predicted to extend at least one density scale height in the internal structure models of Hubbard et al. (1995). We consider it important to first test our hypothesis without the additional complications that arise
due to changes in fluid properties.

Instead, we consider both dynamo and non-magnetic convection models by fixing the magnetic Prandtl number to $Pm = [1, 0]$. This approach allows us to determine the effect of magnetic fields on the convective and zonal flow dynamics. If the flow dynamics are found to be similar between the dynamo and non-magnetic models, electrical conductivity stratification likely has only second-order effects on the hydrodynamics. As a result, our simplified dynamo models may be able to explain the basic global surface observations (zonal flow, thermal emission, magnetic field patterns) of the ice giants. Due to the simplicity of our models, our goal is to reproduce the dynamical patterns, rather than amplitudes. If such Boussinesq models are able to generate ice giant-style dynamical patterns, anelastic models incorporating radially-varying density and electrical conductivity should also be carried out.

We have fixed the model control parameters such that the transition parameter, $Ra/Ra_t = RaE^{7/4}$, exceeds unity, but with a convective Rossby number, $Ro_c = \sqrt{RaE^2/Pr}$, that is formally below unity. This scenario may be relevant to planetary interiors. For example, the zonal Rossby numbers for the ice giants are $Ro_{ZF} \sim 0.1$ (Figure 17), while our models typically have $Ro_{ZF} \sim 0.3$ (Table 9). In all of our models, the control parameters are fixed to $Ra = 2.22 \times 10^7$, $E = 1.5 \times 10^{-4}$, and $Pr = 1$ such that $Ra/Ra_t = 4.5$ and $Ro_c = 0.71$. The simulations are strongly supercritical with $Ra = 110Ra_c$ for the thick shell cases and $Ra = 190Ra_c$ for the thin shell cases. The critical Rayleigh numbers, $Ra_c$, were taken from Al-Shamali
et al. (2004) for non-magnetic rotating convection models with our $E$, $Pr$, and $\chi$ parameters. The dataset for this study is detailed in Table 9.

| $\chi$ | $l_{max}$ | Nu | $Re_c$ | $Re$ | $Ro_{ZF}$ | $C_{\omega z}$ | $|H_{rel}^z|$ | $k_u$ | $k_B$ | f | $\Lambda_i$ | $\Lambda_d$ |
|-------|----------|----|-------|-----|-----------|--------------|------------|------|------|---|----------|----------|
| 0.35  | 192      | 13.4 | 743  | 954 | -0.31     | 0.40         | 0.05       | 5.5  | 26   | 0.16 | 2.0      | 0.03     |
|       |          |      |       |     |           |              |            |      |      |     |          |          |
|       |          |      |       |     |           |              |            |      |      |     |          |          |
| 0.75  | 213      | 24.7 | 859  | 1126| -0.20     | 0.22         | 0.01       | 25   | 92   | 0.04 | 7.1      | 0.18     |
|       |          |      |       |     |           |              |            |      |      |     |          |          |
|       |          |      |       |     |           |              |            |      |      |     |          |          |
|       |          |      |       |     |           |              |            |      |      |     |          |          |
|       |          |      |       |     |           |              |            |      |      |     |          |          |

Table 9: Input and output parameters for fixed $Ra = 2.22 \times 10^7$, $E = 1.5 \times 10^{-4}$, and $Pm = Pr = 1$.

Non-magnetic ($Pm = 0$, $Pr = 1$) values are given in parentheses. The number of radial grid points in the outer and inner cores are $N_{ro} = 65$ and $N_{ri} = 17$, respectively. As is customary in the atmospheric community, the zonal Rossby number, $Ro_{ZF} = U_\phi/\Omega R_O$, assumes the outer shell radius as the length scale, rather than shell thickness, and neglects the factor of two in the denominator. Here, $U_\phi$ is the zonal flow velocity on the outer boundary at the equator; negative indicates retrograde flow. All other parameters are defined in Tables 2 and 5. All quantities are time-averaged, except for $C_{\omega z}$ and $|H_{rel}^z|$, which are instantaneous.

The thick shell simulations use a numerical grid with 192 spherical harmonic modes, 65 radial levels in the fluid outer shell, and 17 radial levels in the solid inner core; the resolution is increased to 213 harmonic modes for the thin shell simulations. No azimuthal symmetries or hyperdiffusivities are employed. Dynamo simulations are initialized using the results of prior dynamo models. Non-magnetic simulations are initialized by changing the electrical conductivity of the model to be insulating and letting the system re-equilibrate, as done in the terrestrial-style survey. All models
are time averaged over at least 0.2 viscous diffusion times once the initial transient behavior has subsided.

It is also beneficial to consider the physical properties of the system in dimensional units as done by Jones and Kuzanyan (2009). Uranus and Neptune both have planetary radii of \( R_P \approx 25,000 \text{ km} \), yielding convecting region thicknesses of \( D = (1 - \chi)R_P \approx 16,000 \text{ km} \) for the thick shell models and \( D \approx 6300 \text{ km} \) for the thin shell models. In addition, the rotation rate is approximately \( \Omega \approx 10^{-4} \text{ s}^{-1} \) for both planets. Since \( E = 1.5 \times 10^{-4} \) and \( Pr = Pm = 1 \) are fixed in our dynamo models, the diffusivities are approximately \( \nu = \kappa = \eta \approx 8 \times 10^6 \text{ m}^2/\text{s} \) and \( \nu = \kappa = \eta \approx 1 \times 10^6 \text{ m}^2/\text{s} \) in the thick and thin shell models, respectively. In contrast, the diffusivity estimates for the ice giants are \( \nu \sim 10^{-6} \text{ m}^2/\text{s}, \kappa \sim 10^{-7} \text{ m}^2/\text{s}, \) and \( \eta \sim 10^2 \text{ m}^2/\text{s} \). The simulated diffusivities are, therefore, many orders of magnitude larger than the planetary values.

For the thick and thin models, respectively, the velocity scale is given by \( \nu/D \sim 0.5 \text{ m/s} \) and \( 0.2 \text{ m/s} \). Typical dimensionless velocities are \( Re \sim 10^3 \) such that the typical dimensional velocity is 500 m/s for the Neptune-like model and 200 m/s for the Uranus-like model. These estimates are of the same order as the observed winds.

In a similar manner, the magnetic induction scale is given by \( \sqrt{2\rho \mu_0 \eta \Omega} \sim 1 \text{ T} \) in both of our dynamo models. Since the typical dimensionless magnetic field strengths, \( \sqrt{\Lambda_i} \), are approximately unity, the typical dimensional magnetic field strengths are roughly 1 T in our models. These estimates are significantly stronger than the measured
strengths of about $\sim 10 \mu \text{T}$ (Table 1). This overestimation likely occurs because the magnetic field strength scales with $Pm$ and with the convective flux of the system, both of which are orders of magnitude larger in the models than in the planets (Christensen and Aubert, 2006).

Following an approach similar to Jones and Kuzanyan (2009), the dimensional heat flux scale can be written as

$$Q = Ra \frac{\nu \kappa^2 \rho C_p}{\alpha g D^4}. \quad (51)$$

For $Ra = 2.22 \times 10^7$, $\rho \sim 10^3 \text{ kg/m}^3$, $C_p \sim 10^3 \text{ J/kg K}$, $\alpha \sim 10^{-3} \text{ K}^{-1}$, and $g \sim 10 \text{ m/s}^2$ with the diffusivities given above, the unit of heat flux is approximately $10^7 \text{ W/m}^2$ in the thick shell models and $10^6 \text{ W/m}^2$ in the thin shell models. Our simulations typically have dimensionless heat fluxes of $\frac{dT}{dr} \frac{D}{\Delta T} \sim \mathcal{O}(10)$. In contrast, the emitted flux of Neptune is 0.4 W/m$^2$ and is 0.04 W/m$^2$ for Uranus (Pearl and Conrath, 1991; Pearl et al., 1990). Enhanced heat flux is necessary to drive convection and dynamo action in all present day dynamo models because of artificially large viscosities (Christensen, 2010).

Let us instead consider the heat flux assuming turbulent diffusivities where $\nu_T \sim \kappa_T \sim 10 \text{ m}^2/\text{s}$ such that $Pr_T \sim 1$, $E_T \sim 10^{-10}$ for the thick shell model, and $E_T \sim 10^{-9}$ for the thin shell model (cf. Starchenko and Jones, 2002; Jones and Kuzanyan, 2009). In order to fix the transition parameter to the value used in our simulations, $Ra/Ra_t = RaE^{7/4} = 4.5$, we consider $Ra \sim 10^{18}$ in the thick shell model and $Ra \sim 10^{16}$ in the thin shell model. Using these estimates for $Ra$ and the turbulent diffusivities,
the heat flux scaling yields \( Q \sim 1.5 \text{ W/m}^2 \) and \( Q \sim 0.6 \text{ W/m}^2 \) in the thick and thin shell models, respectively. These estimates are in better agreement with the internal planetary heat fluxes \( (Q_{\text{Neptune}} \sim 0.4 \text{ W/m}^2; Q_{\text{Uranus}} \sim 0.04 \text{ W/m}^2) \). Since the simulated and planetary heat flux disparity decreases as more extreme parameters are considered, inertial convection in the ice giants may be energetically feasible.

4.5 Flow Characteristics

The characteristics of convection, zonal flows, and meridional circulations are discussed below.

4.5.1 Energetics

Figure 21 shows the kinetic energies split into components. Following the conventions of the solar dynamo literature (e.g., Brown et al., 2010), kinetic energy consists of zonal flow energy due to axisymmetric toroidal motions, meridional circulation energy due to axisymmetric poloidal motions, and convective energy due to non-axisymmetric motions. More than half of the kinetic energy in the dynamo models is due to non-axisymmetric convective fluid motions, with the remaining energy contained in the zonal flows. Meridional circulations contribute less than 1% to the kinetic energy in all of our models. In contrast, the non-magnetic models contain 60% and 90% of the kinetic energy in the zonal flows in the thick and thin shells,
respectively. Consequently, magnetic fields weaken the zonal flow contribution to kinetic energy.

Figure 21: Decomposition of the kinetic energies. Here, $ZF$ is zonal flow kinetic energy, $MC$ is meridional circulation kinetic energy, $Conv$ is non-axisymmetric convective kinetic energy. Kinetic energy densities, $\mathcal{E}_K$, are also given.

4.5.2 Structure of Convection, Zonal Flows, and Meridional Circulations

The structure of convection is indicated by axial vorticity columnarity and relative axial helicity, which are defined in Table 5 and given in Table 9. The non-axisymmetric convective motions in all of our models are three-dimensional with
$C_{\omega z} \gtrsim 0.4$. These unstructured motions are poorly correlated and helicity is small with $|H_z^{rel}| < 0.1$.

Figure 22: Comparison of observed and simulated zonal flows at the outer shell boundary for a) Neptune and our thick shell models and b) Uranus and our thin shell models. Colored solid lines indicate dynamo models, and colored dashed lines indicate non-magnetic models. The black long-dashed (short-dashed) lines denote observed profiles assuming Voyager 2 (Helled et al., 2010) rotation rates. Velocity is given in zonal Rossby number units.

Figure 22 compares the observed and simulated zonal flows. This comparison shows that, regardless of the presence of magnetic fields and differences in geometry, the models are characterized by retrograde equatorial jets, in qualitative agreement with the observations. While magnetic fields and thinner fluid shells both cause the amplitude of the equatorial jet to decrease, the flow speeds in all of our models are significantly stronger than those of the ice giants. In further qualitative agreement
with observations, prograde winds also develop at high latitudes in all of our models except for the thin shell dynamo. Here, magnetic fields strongly damp the zonal flows and weak prograde and retrograde jets develop instead. We, therefore, conclude that inertial convection leads to ice giant-style zonal flows and infer that magnetic field effects on the surface zonal flows must be relatively weak so as not to fundamentally alter the high latitude jets.

Figure 23 illustrates that the zonal flows also behave similarly at depth in our models since they do not exhibit significant variations in the axial direction; the simulated meridional circulations are also compared. Two large circulation cells develop outside of the tangent cylinder in all of our models. These cells circulate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere, resulting in an equatorial upwelling and downwellings at mid-latitudes. In the thick shell models, each cell consists of two smaller circulations with the same orientation; these circulations are separated at the cylindrical radius where the zonal flow velocity transitions between the prograde and retrograde directions. Cells also develop above and below the inner core with opposite polarities in the thick shell geometry. These polar cells promote the mid-latitude downwellings and lead to upwellings near both poles. Meridional circulations are weak within the tangent cylinder of the thin shell models.
Figure 23: Time-averaged meridional circulations (left panels) and zonal flows (right panels). Red (blue) indicates both clockwise (counterclockwise) meridional circulations and prograde (retrograde) zonal motions. Flow speeds are given in zonal Rossby number units, $R_{ZF} = U_\phi/\Omega R_O$. 
4.5.3 Angular Momentum Mixing

Aurnou et al. (2007) argue that retrograde equatorial jets are the result of absolute angular momentum homogenization. Absolute angular momentum is the angular momentum of a parcel of fluid measured in the inertial reference frame and is defined as 

\[ M = \rho u_\phi s + \rho \Omega s^2, \]

where \( s \) is cylindrical radius. The first and second terms are the angular momentum due to zonal flows and solid body rotation, respectively. When \( M \) is non-dimensionalized by the angular momentum of a parcel on the outer shell boundary moving in solid body rotation, this equation can be re-arranged to give

\[ Ro_{ZF}(s) = \frac{R_O}{s} M^* - \frac{s}{R_O}, \]

yielding a prediction for the zonal flow profile. Here, \( Ro_{ZF} = U_\phi/\Omega R_O \) and \( M^* = M/\rho \Omega R_O^2 \). Because angular momentum is conserved and the system initially assumes solid body rotation, \( M^* \) can be determined by

\[ M^* = \frac{1}{V} \int_V \frac{\rho \Omega s^2}{\rho \Omega R_O^2} dV = \frac{1}{\frac{4}{3} \pi (R_O^3 - R_I^3)} \int_0^{2\pi} \int_0^\pi \int_{R_I}^{R_O} \frac{r^4 \sin^3 \theta}{R_O^2} dr \, d\theta \, d\phi = \frac{2}{5} \left( \frac{1 - \chi^5}{1 - \chi^3} \right), \]

This yields \( M^* = [0.42, 0.53] \) for \( \chi = R_I/R_O = [0.35, 0.75] \).

Angular momentum homogenization then predicts zonal flows to vary only with distance from the rotation axis where retrograde (prograde) flows develop far from (near) the rotation axis. Further, equatorial zonal wind speeds are expected to decrease with decreased shell thickness and when less of the fluid volume is well-mixed.
Figure 24: a) Dimensionless absolute angular momentum on the outer shell boundaries. Mixing of absolute angular momentum is limited to within about 40° from the equator. Comparison of time-averaged zonal flow profiles to those predicted by (52) when mixing occurs in the entire shell and in regions exterior to 40° latitude in the b) thick shell models and c) thin shell models.

Figure 24a plots the dimensionless absolute angular momentum, $M^*$, on the outer shell boundary as a function of latitude. All profiles show that $M^*$ is approximately constant within about ±40° latitude for most models, consistent with absolute angular momentum mixing.

The simulated zonal winds are compared against the profiles predicted by (52) in Figure 24 b,c. Good agreement between the simulated and predicted profiles occurs at low latitudes. At high latitudes, however, the profiles diverge because the predicted flows approach infinity near the poles. Instead, the winds are damped with no flow at the poles. Strong magnetic damping is also evident in the thin shell dynamo model.
The reversal in flow direction of the high latitude zonal jet is clearly associated with magnetic fields, but the mechanism responsible is not currently well understood.

4.5.4 Magnetic Damping

Figure 22a clearly shows that magnetic fields act to reduce the zonal flow speeds. This can be seen by considering the zonal power budget, which illuminates the sources and sinks of zonal energy. This budget is obtained by multiplying the axisymmetric azimuthal component of the momentum equation (41) by the zonal flow, \([u_\phi]\):

\[
\frac{1}{2} \frac{\partial}{\partial t} [u_\phi]^2 = -Ro [u_\phi][u \cdot \nabla u]_\phi - [u_\phi][u_s] + E_i [u_\phi][\nabla^2 u]_\phi + \Lambda_d [u_\phi] [((\nabla \times B) \times B)]_\phi. \tag{54}
\]

Here, square brackets indicate azimuthal averaging and subscripts indicate the vector component. The first term on the right side generates zonal flows through advection, and the second term represents the exchange of power between zonal flows and meridional circulations via the Coriolis force. The remaining terms tend to restore the system to solid body rotation through viscous and magnetic damping, respectively.

For simplicity, let us consider an azimuthal flow occurring on cylinders, \(U_\phi(s) \hat{\phi}\), subject to a general background flow. The zonal budget equation then simplifies to

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} U_\phi^2 \right) = -Ro U_\phi [u \cdot \nabla u]_\phi + E_i U_\phi [\nabla^2 u]_\phi + \Lambda_d U_\phi [((\nabla \times B) \times B)]_\phi. \tag{55}
\]

Note that the Lorentz term is dissipative because Lenz’s law dictates that the magnetic fields generated via zonal flow shearing due to the \(\Omega\)-effect will oppose the force
that increased the magnetic tension (e.g., Aubert, 2005). Magnetic fields are then expected to strongly damp the zonal flows when the ratio of zonal Lorentz to zonal viscous forces exceeds unity. This zonal force ratio is calculated to be 0.6 in the thick shell dynamo model and 3.4 in the thin shell dynamo using the approach outlined in Section 3.4.2. Thus, the zonal flows are similar between the dynamo and non-magnetic ice giant-style models in the thick shell geometry because viscous damping exceeds magnetic damping, while the zonal flows differ more strongly in the thin shell geometry because magnetic damping exceeds viscous damping.

4.6 Heat Flux Patterns

The pattern of outer boundary heat flux in our modes is predominantly set via the mean convective heat flux, \( \langle V_r T \rangle \) where angled brackets indicate azimuthal and temporal averaging (e.g. Kundu and Cohen, 2002). Figures 25 and 26 show the mean radial convective heat flux profiles in our thick and thin shell models, respectively (e.g. Kundu and Cohen, 2002). Regions of enhanced temperature correlate well with upwelling flows and lead to positive convective heat fluxes, while regions of diminished temperature correlate with downwellings flows and lead to negative convective heat fluxes. Consequently, meridional circulations in the thick shell models lead to peak heat flux along the outer boundary near the poles and equator. As shown in panel 25d, this result holds regardless of the presence of magnetic fields in these models.
Panel 25e tests how well the mean radial convective fluxes explain the thermal emission patterns by comparing the differential convective flux between the dynamo and non-magnetic models (panel 25c) averaged over the outermost one quarter of the shell against the differential outer boundary heat flux between the dynamo and non-magnetic models (panel 25d). These profiles tend to be similar, although we note that the behaviors differ near ±60° latitude. This similarity implies that the pattern of outer boundary heat flux is controlled primarily by the mean, as opposed to turbulent, convective fluxes. Further, these changes in mean radial convective fluxes can then explain the decreased equatorial flux and enhanced high latitude flux along the outer boundary in the non-magnetic model.

Equatorial upwelling also develops in both thin shell models and leads to enhanced mean radial convective fluxes and outer boundary heat fluxes at low latitudes (Figure 26). This enhancement is stronger in the non-magnetic model because the mean radial velocities are faster. At high latitudes, however, there is a notable difference in heat transfer behavior between the dynamo and non-magnetic thin shell models. This difference is likely due to fluctuating radial convective fluxes, which were not output during the production runs.
Figure 25: Mean radial convective heat flux profiles for the thick shell a) dynamo and b) non-magnetic models; c) shows the difference between the two flux profiles. d) Dimensionless heat flux profiles on the outer boundary, non-dimensionalized by $D/\Delta T$. e) Comparison of the difference in mean convective fluxes averaged over the outermost one quarter of the fluid between the dynamo and non-magnetic models against the difference in outer boundary heat fluxes with and without magnetic fields.
Figure 26: Mean radial convective heat flux profiles for the thin shell a) dynamo and b) non-magnetic models; c) shows the difference between the two flux profiles. d) Dimensionless heat flux profiles on the outer boundary, non-dimensionalized by $D/\Delta T$. e) Comparison of the difference in mean convective fluxes averaged over the outermost one quarter of the fluid between the dynamo and non-magnetic models against the difference in outer boundary heat fluxes with and without magnetic fields.
4.7 Magnetic Field Generation

Magnetic field generation can occur on both local and global scales (Cattaneo, 1999). Mean-field dynamos result when the fluid motions promote field generation at scales larger than that of the convection. Zonal flows shear poloidal magnetic fields to form toroidal magnetic fields via the Ω-effect, and helical motions create poloidal magnetic fields by twisting toroidal field lines via the α-effect (e.g., Moffatt, 1978). In contrast, fluctuating dynamos result when fluid motions randomly stretch and amplify magnetic fields at scales smaller than that of the turbulence (e.g., Schekochihin et al., 2008). Thus, the dominant regime of dynamo generation can be determined by contrasting the magnetic and velocity length scales.

Typical length scales of the magnetic and flow fields are quantified using the characteristic wavenumbers, $k_u$ and $k_B$, defined in Table 5 and given in Table 9. Regardless of shell geometry, we find that the magnetic field is smaller scale than the velocity field ($k_B > 3k_u$) in the dynamo models. This scale disparity suggests that the magnetic fields in our models are generated by small-scale, fluctuating dynamo action.

The radial magnetic and velocity fields near the outer shell boundaries of the dynamo models at full resolution are shown in Figure 27. This visual comparison illustrates the difference in spatial scale between the magnetic and velocity fields, where the magnetic flux patches are smaller in scale than the inflow and outflow structures. The magnetic fields are characterized by locally strong flux patches where patches of
the thin shell model are much smaller in scale and larger in amplitude than those of
the thick shell model. Further, regions of intense magnetic fields tend to coincide with
strong radially-inward flows. This occurs because the fluid converges at these regions
and collects the magnetic field lines (e.g., Olson et al., 1999). These poorly-organized,
turbulent flow and field structures are consistent with our fluctuating dynamo argu-
ment.

Figure 27: Snapshots of the radial magnetic and velocity fields near the outer boundary. The
color scheme for radial velocity has been skewed to emphasize the downwelling regions, which tend
to correlate with regions of strong magnetic fields.
4.8 Comparison of Model Results Against Observations

Figure 28 compares the thick shell dynamo model against the Neptunian observations shown in Figure 17a. The model self-consistently generates zonal flow, thermal emission, and magnetic field patterns that are similar to those of Neptune: a retrograde zonal jet at low latitudes and prograde jets at high latitudes, peak thermal emission in the equatorial and polar regions, and a multipolar dynamo. The zonal flow pattern does not vary strongly in time as shown by the comparison of time-averaged and instantaneous profiles. In contrast, a similar comparison of heat flux profiles shows that the thermal emission evolves more strongly with time. The magnetic field also exhibits significant temporal variations, but remains multipolar when averaged over time as shown by the magnetic power spectra (Figure 30). It is possible for instantaneous heat flux and magnetic field morphologies to be in good agreement with the observations; however, we chose to use random snapshots in time rather than tuning our results.
<table>
<thead>
<tr>
<th>Zonal Flows</th>
<th>Thermal Emissions</th>
<th>Magnetic Fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zonal flow, m/s</td>
<td>Heat Flux, W/m²</td>
<td>1 bar Radial Field Br (l,m≤3)</td>
</tr>
<tr>
<td>Zonal Flow, Ro₉₂=U/q₁R₀</td>
<td>Dimensionless Heat Flux</td>
<td>[10⁻², 10¹]</td>
</tr>
</tbody>
</table>

**Figure 28:** a) Neptunian observations compared against b) simulated zonal flows, heat fluxes, and radial magnetic fields (l, m ≤ 3) for the thick shell model. The zonal flows and heat fluxes in bold are time-averaged, while the lighter lines are random snapshots in time. Magnetic fields are instantaneous. The dimensionless planetary thermal emission profiles have been normalized such that the maximum value is unity; the dimensionless simulated thermal emission profiles have been non-dimensionalized by D/∆T.
Figure 29 compares the thin shell dynamo model against the Uranian observations shown in Figure 17b. The model generates several important aspects of ice giant-style dynamics: a retrograde equatorial zonal jet and a multipolar magnetic field morphology. A comparison of thermal emission patterns is less meaningful given that the observed profile is poorly constrained. An importance deviation of the model compared to the observations is the lack of strong prograde zonal jets at high latitudes, which are magnetically damped in the model. We predict, however, that if a thin electrically insulating layer were present near the outer boundary, prograde jets may develop near the surface at high latitudes, bringing the zonal flow pattern into qualitative agreement with the observations. As found for the thick shell dynamo model, the zonal flows are quasi-steady, while the thermal emissions and magnetic fields are temporally variable.
<table>
<thead>
<tr>
<th>Zonal Flows</th>
<th>Thermal Emissions</th>
<th>Magnetic Fields</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Zonal Flow, m/s</strong></td>
<td><strong>Heat Flux, W/m²</strong></td>
<td>**1 bar Radial Field Br (</td>
</tr>
<tr>
<td>-500 0 500</td>
<td>-500 0 500</td>
<td>-200 mT 0 μT 200 mT</td>
</tr>
<tr>
<td>Zonal Flow, ( R_{o} \rho = \frac{U}{\Omega R_{o}} )</td>
<td>Dimensionless Heat Flux</td>
<td>( \Lambda = B/2\mu_{B}\rho \Omega = 0 ) to ( 10^{4} )</td>
</tr>
<tr>
<td>Latitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-80 0 80</td>
<td>-100 0 100</td>
<td></td>
</tr>
<tr>
<td>-60 0 60</td>
<td>-80 0 80</td>
<td></td>
</tr>
<tr>
<td>-40 0 40</td>
<td>-60 0 60</td>
<td></td>
</tr>
<tr>
<td>-20 0 20</td>
<td>-40 0 40</td>
<td></td>
</tr>
<tr>
<td>0 0 0</td>
<td>0 0 0</td>
<td></td>
</tr>
</tbody>
</table>

Figure 29: **a)** Uranian observations compared against **b)** simulated zonal flows, heat fluxes, and radial magnetic fields \( (l, m \leq 3) \) for the thin shell model. The zonal flows and heat fluxes in bold are time-averaged, while the lighter lines are random snapshots in time. Magnetic fields are instantaneous. The dimensionless planetary thermal emission profiles have been normalized such that the maximum value is unity; the dimensionless simulated thermal emission profiles have been non-dimensionalized by \( D/\Delta T \).
We also contrast the simulated amplitudes against the observations as done in Section 4.4 following the approach of Jones and Kuzanyan (2009) to obtain dimensional values of the physical properties. The thick (thin) shell dynamo simulation has a peak equatorial wind speed of about 800 m/s (500 m/s). Given the possible range in wind speeds due to uncertainties in the planets’ rotation rates, this difference of up to a factor of five may be considered acceptable.

In addition, the rms magnetic field strength is about 100 mT (200 mT) in the thick (thin) shell dynamo models when truncated at spherical harmonic degree $l = 3$ to match the resolution limit of the observations. Both simulated magnetic fields are then about two orders of magnitude stronger than the observed fields. This overestimation may be reduced as lower magnetic Prandtl numbers are considered since the magnetic field strength appears to scale with $Pm$ (Christensen and Aubert, 2006; Gomez-Perez and Heimpel, 2007).

The simulated heat flux is also many orders of magnitude larger than the observed heat flux. However, as discussed in Section 4.4, no overdriving may be necessary when the results are scaled to low Ekman numbers.
Figure 30: a) and b) Comparison of (instantaneous) observed versus time-averaged (bold black symbols) and instantaneous (lighter black symbols) simulated magnetic power spectra up to spherical harmonic degree and order three. d) and c) Comparison of time-averaged magnetic power spectra in our models and in Model 1 of Stanley and Bloxham (2006). The left panels have been normalized by the dipolar ($l = 1$) power, and the right panels have been normalized by the axisymmetric ($m = 0$) power. The simulated spectra are taken at the outer shell boundary and have not been upward continued.
The observed magnetic spectra are compared against time-averaged and instantaneous simulated spectra in Figure 30 a,b. Both planets and models have significant power in the dipole, quadrupole, and octupole components. The planets have a strong $m = 1$ signature and much weaker power at higher orders. The randomly chosen instantaneous spectra are non-axisymmetric, yet both models have peak power in the axisymmetric $m = 0$ mode when temporally averaged.

Figure 30 c,d compares the time-averaged magnetic spectra of our models and Model 1 of Stanley and Bloxham (2006). This comparison shows that our models fit the observations approximately as well as the model of Stanley and Bloxham (2006) – while simultaneously generating zonal flow and thermal emission patterns that are roughly similar to those observed. No other models, to our knowledge, are able to self-consistently produce these dynamical characteristics of the ice giants.

We utilize the Boussinesq approximation, where electrical conductivity and density stratification are neglected, and must assess the consequences of this limitation. A comparison between dynamo and otherwise identical non-magnetic models has shown that magnetic fields can substantially decrease the zonal flow speeds. Further, in the thin shell dynamo model, magnetic effects cause the direction of the high latitude zonal jet to reverse. Thus, magnetic field effects may be important and merit further investigation. Chapter 5 addresses the influence of changing the electrical conductivity of the fluid in strongly-forced models.

In addition, our zonal flows and meridional circulations are qualitatively similar to
those obtained in the slowly-rotating anelastic models of Brun and Palacios (2009), where the density difference between the upper and lower boundaries is two orders of magnitude (about five density scale heights). Since we have estimated that the dynamo regions of Uranus and Neptune contain only about one density scale height, our agreement with Brun and Palacios (2009) suggests that density stratification will not fundamentally alter our results.

4.9 Conclusions

The ice giants, Uranus and Neptune, are unique in the solar system. Observations show that the planets have zonal flows, thermal emissions, and magnetic fields that are fundamentally different from those of Jupiter and Saturn. The winds and magnetic fields are thought to be the result of convection and dynamo action in the planetary interior, which will also affect the thermal emissions. If the magnetic fields are driven near the surface, a single region may simultaneously generate these observables.

Towards testing the hypothesis that inertial convection controls the generation of ice giant-style dynamics (Aurnou et al., 2007), we carry out strongly-forced dynamo and non-magnetic models with two spherical shell geometries. In these simulations, zonal flows characterized by retrograde equatorial jets result from the homogenization of absolute angular momentum mixing via three-dimensional turbulence. Meridional circulations are coupled with the transport of heat and lead to peak heat flux near the
equator and additionally near the poles in the thick shell geometry. The magnetic fields in our models are generated by fluctuating dynamos due to the small-scale, poorly-organized nature of the convection. We further show that multipolar dynamos occur irrespective of the shell geometry. This result sharply differs from the Stanley and Bloxham (2004, 2006) hypothesis that convective region geometry dictates the generation of multipolar dynamos.

Our dynamo models self-consistently generate zonal flow, thermal emission, and magnetic field patterns that are roughly comparable to those of Uranus and Neptune. Thus, our models are consistent with the hypothesized difference between the gas and ice giants. This work then suggests that inertially-dominated anelastic dynamo models with radially-varying density and electrical conductivity should be carried out in order to further test this hypothesis.
5 Varying the Magnetic Prandtl Number in Inertial Dynamos

5.1 Introduction

The previous chapter studied models in the “inertial” convective regime. Parallel to most rapidly-rotating dynamo studies, we fixed \( Pr = Pm = 1 \). However, this typical \( Pm \) value significantly differs from that of planetary dynamos, where the molecular magnetic Prandtl number values are on the order of \( Pm \lesssim 10^{-6} \) (see Table 4) and turbulent \( Pm \) values are expected to be no greater than \( O(10^{-2}) \) (e.g., Roberts and Aurnou, 2011; Busse and Simitev, 2011). Although not well tested, it is often assumed in the literature that no fundamental changes occur in the flow or magnetic fields as model results are extrapolated to low magnetic Prandtl numbers since the simulated and planetary magnetic Reynolds numbers tend to be comparable (e.g., Christensen and Wicht, 2007). Planetary magnetic Reynolds numbers estimates typically range between \( 10^2 \lesssim Rm \lesssim 10^3 \) (e.g., Bloxham and Jackson, 1991; Starchenko and Jones, 2002).

It is difficult to carry out numerical dynamo models with \( Pm \ll 1 \) because turbulence \((Re \gg 1)\) is required in order to maintain dynamo action: \( Rm = RePm \gtrsim 10^2 \). Since \( Re = Ro/E \), high Reynolds number flows can be achieved in rapidly-rotating models \((Ro \ll 1)\) at low Ekman numbers (e.g., Schaeffer and Cardin, 2006) or in strongly-
forced models \((Ro \gtrsim 1)\) at moderate Ekman numbers.

Despite these difficulties, low magnetic Prandtl number dynamics have previously been investigated in the rotationally-dominated convective regime, where the convective flow field tends to be characterized by columnar structures (Simitev and Busse, 2005; Sreenivasan and Jones, 2006; Gomez-Perez and Heimpel, 2007). Simitev and Busse (2005) show that the magnetic field morphology changes when \(Pm\) is varied while the other control parameters are kept at fixed values (e.g., \(E = 10^{-5}, Pr = 1\)). Particularly, large quadrupolar components occur when the magnetic Prandtl number is near its onset value. As \(Pm\) is increased, the field morphology transitions to hemispherical then to axially-aligned dipoles and then to chaotic dynamos.

Similarly, Sreenivasan and Jones (2006) vary both \(Pr\) and \(Pm\) while maintaining a constant ratio of thermal to magnetic diffusivities. For rapidly-rotating models with \(Pm = Pr \geq 1\), the magnetic fields are dipole-dominated and the flow speeds and structures are only weakly affected by diffusivity changes. In contrast, the magnetic field weakens, becomes less dipolar, and the flow field becomes more irregular in models with \(Pm = Pr < 1\) due to the increased effect of inertia.

In addition, Gomez-Perez and Heimpel (2007) vary \(Pm\) and consider a range of thermal forcings from near onset to moderately supercritical in the rapidly rotating regime. As the magnetic Prandtl number is decreased and/or the thermal forcing is increased, the amplitude of the zonal flow increases and the dipole-component of the magnetic field becomes weaker.
Little work, however, has been done to investigate the influence of $Pm$ in the inertially-dominated regime. Towards testing the assumption that the magnetic Reynolds number, rather than the magnetic Prandtl number, controls the applicability of model results to planetary settings, we vary the magnetic Prandtl number in a strongly-forced model to measure how the dynamics systematically change with $Pm$ in the inertial regime for a fixed thermal forcing ($Ra$). We find that the fluid motions, pattern of convective heat transfer, and mode of dynamo generation all differ across $Pm$ space. For example, we show that strong magnetic fields cause a fundamental change in the surface zonal flows: the equatorial zonal jet reverses as the magnetic Prandtl number is increased. However, the magnetic Reynolds number is also shown to vary across the survey. As such, we plan to carry out additional simulations where $Rm$, rather than $Ra$, is fixed for a range of $Pm$ values. The numerical survey is described in Section 5.2, behavioral regimes are discussed in Section 5.3, and regime transitions are examine in Section 5.4. Our conclusions are given in Section 5.6.

5.2 Numerical Model

Our strongly-forced survey consists of seven dynamo models and a non-magnetic model for comparison. The control parameters are set to $\chi = 0.35$, $E = 1.5 \times 10^{-4}$, $Pr = 1$, and $Ra = 2.22 \times 10^7$ such that $Ra/Ra_t = 4.5$ as done in the models of Chapter 4. For simplicity and to reduce the significant computational expenditure required for these simulations, we consider only a thick spherical shell geometry here.
The magnetic Prandtl number is varied over the range \(0.25 \leq Pm \leq 10\). This dataset is detailed in Table 10.

The simulations use a numerical grid with 192 spherical harmonic modes, 65 radial levels in the outer shell, and 17 radial levels in the inner core. No azimuthal symmetries or hyperdiffusivities are employed. The simulations are initialized using the results of prior dynamo models. All models are time averaged over at least 0.2 viscous diffusion times once the initial transient behavior has subsided.

| \(Pm\) | \(Rm\) | \(Re\) | \(Re_c\) | \(RoZF\) | \(Nu\) | \(C_{wz}\) | \(|H_{z}^{rel}|\) | \(f_{I}\) | \(q_{I}\) | \(\theta_{dip}\) | \(\Lambda_{i}\) | \(\Lambda_{d}\) |
|-------|-------|-------|--------|---------|------|---------|----------------|-----|-----|-------------|------|------|
| 10    | 4703  | 470   | 392    | +0.13   | 12.3 | 0.31    | 0.06           | 14  | 51  | 0.03        | 0.04 | 40   | 57   | 0.39  |
| 5     | 2614  | 523   | 402    | +0.16   | 12.5 | 0.29    | 0.06           | 12  | 46  | 0.04        | 0.05 | 45   | 23   | 0.26  |
| 2     | 1077  | 538   | 464    | +0.11   | 12.7 | 0.31    | 0.08           | 13  | 37  | 0.04        | 0.06 | 49   | 4.5  | 0.10  |
| 1     | 954   | 954   | 743    | -0.31   | 13.4 | 0.40    | 0.05           | 5.5 | 26  | 0.18        | 0.18 | 56   | 2.0  | 0.03  |
| 0.75  | 740   | 987   | 744    | -0.33   | 13.4 | 0.33    | 0.06           | 5.4 | 24  | 0.17        | 0.20 | 57   | 1.1  | 0.02  |
| 0.50  | 504   | 1009  | 721    | -0.34   | 13.2 | 0.35    | 0.04           | 5.4 | 19  | 0.23        | 0.18 | 58   | 0.57 | 0.01  |
| 0.25  | 266   | 1065  | 678    | -0.38   | 13.0 | 0.32    | 0.04           | 5.4 | 15  | 0.13        | 0.59 | 61   | 0.09 | 0.003 |
| 0     | 1097  | 693   | -0.37  | 13.0    | 0.31 | 0.06    | 5.2            |     |     |             |      |      |      |      |

Table 10: Input and output parameters for fixed \(\chi = 0.35\), \(Ra = 2.22 \times 10^7\), \(E = 1.5 \times 10^{-4}\), and \(Pr = 1\). Parameters are defined in Tables 2 and 5. \(RoZF = U_{\phi}/\Omega R_O\) assumes the zonal flow velocity on the outer boundary at the equator; negative (positive) indicates retrograde (prograde) flow. \(\theta_{dip}\) is the mean latitude of the dipole over time. All quantities are time-averaged, except for \(C_{wz}\) and \(|H_{z}^{rel}|\) which are instantaneous. Models are in the SMP Regime (strongly multipolar, prograde) above the short-dashed line, in the AMP Regime (axial multipolar, retrograde) between the dashed lines, and in the AQ Regime (axial quadrupolar, retrograde) below the long-dashed line.
5.3 Behavioral Regimes

Figure 31 identifies three behavioral regimes in our suite of strongly-forced dynamo models based on magnetic field morphology and zonal flow patterns.

The magnetic fields change in morphology as the magnetic Prandtl number is varied. The magnetic field morphology is quantified by its internal dipolarity,

\[
f_I = \left( \frac{\int B_{l=1} \cdot B_{l=1} \, dV}{\int B \cdot B \, dV} \right)^{1/2},
\]

and its internal quadrupolarity,

\[
q_I = \left( \frac{\int B_{l=2} \cdot B_{l=2} \, dV}{\int B \cdot B \, dV} \right)^{1/2}.
\]

Here, \(B_{l=1}\) is the dipole field strength, \(B_{l=2}\) is the quadrupole field strength, \(B\) is the total field strength, and \(\int dV\) integrates over the volume of the spherical fluid shell.

We consider cases with \(f_I \gtrsim 0.5\) to be dipole-dominated and cases with \(q_I \gtrsim 0.5\) to be quadrupole-dominated. These values, shown in Figure 31a and given in Table 10, indicate that the magnetic energy is spread over a range of harmonic degrees (\(f_I \sim q_I \lesssim 0.2\)) in all models except for the AQ model, where the field is dominated by the quadrupolar component with \(q_I = 0.59\).
Figure 31:  

a) Time-averaged internal dipolarity, $f_I$, and quadrupolarity, $q_I$, versus the magnetic Prandtl number. 

b) Temporally- and azimuthally- averaged zonal Rossby number, $Ro_Z = U_\phi/\Omega R_O$ assumes the zonal flow velocity on the outer boundary at the equator; negative (positive) indicates retrograde (prograde) flow. The three behavioral regimes and transitions are also highlighted.
A fundamental change in zonal flow also occurs in our survey. The zonal Rossby number, $Ro_{ZF} = U_\phi/\Omega R_O$, measures the zonal flow velocity relative to solid body rotation at the outer boundary, $R_O$. This parameter assumes the outer shell radius as the length scale, rather than shell thickness, and neglects the factor of two in the denominator as is customary in the atmospheric community. Here, $U_\phi$ is the zonal flow velocity on the outer boundary at the equator and negative (positive) indicates retrograde (prograde) flow. Figure 31b shows that the direction of the equatorial jet reverses as the magnetic Prandtl number is varied. The jet is retrograde (westward) at low $Pm$ values and prograde (eastward) at high $Pm$ values.

Models with $Pm \geq 2$ are characterized by strongly multipolar (SMP) dynamos and prograde equatorial jets, while models with $0.5 \leq Pm \leq 1$ are characterized by axial multipolar (AMP) dynamos and retrograde equatorial jets. The $Pm = 0.25$ model is characterized by an axial quadrupolar (AQ) dynamo and a retrograde equatorial jet. Dynamo action becomes subcritical near $Pm = 0.2$. These regimes are discussed in more detail below.

### 5.3.1 Magnetic Field Characteristics

Figure 32 shows the magnetic power spectra at full spectral resolution. The simulations are thought to have adequate spatial resolution since there is at least an order of magnitude difference between the maximum and the cut-off power (Christensen et al., 1999).
Figure 32: Time-averaged magnetic field spectra shown at full resolution.
Figure 33: Time-averaged magnetic field spectra up to degree and order 10. a) and c) show the raw data, while b) and d) show the normalized data.
Figure 33 focuses on the spectra up to spherical harmonic degree 10. These spectra show that the SMP dynamos ($Pm \geq 2$) have significant power over a broad range of spherical harmonic degrees and a weak peak at spherical harmonic order $m = 1$. In contrast, the AMP dynamos ($0.5 \leq Pm \leq 1$) have significant axial dipole through quadrupole components, while the $Pm = 0.25$ dynamo is dominated by the axial quadrupole component.

In order to illustrate the magnetic field structures in each morphology regime, the top row of Figure 34 shows the instantaneous axisymmetric magnetic fields for the $Pm = 10$, 1, and 0.25 models. The corresponding instantaneous toroidal magnetic fields in the equatorial planes are given in the bottom row. These snapshots indicate that the magnetic fields become smaller scale as the magnetic Prandtl number is increased. Both toroidal and poloidal magnetic field components are characterized by relatively large-scale flux bundles in the AMP and AQ models (subplots a and b), while the magnetic fields are poorly organized in the SMP models (subplot c).

Figure 35 shows snapshots of the radial magnetic fields near the outer shell boundary for the same models to further visualize the dynamo behaviors. Here, the models clearly indicate that the magnetic field transitions from a larger-scale, hemispherical dynamo to an unstructured, small-scale dynamo with increased magnetic Prandtl number.
Figure 34: Top panel: Instantaneous axisymmetric poloidal (right panel) and toroidal (left panel) magnetic fields for select models. Bottom panel: Instantaneous toroidal magnetic field in the equatorial plane for the same models. Field strength is given in $\sqrt{\Lambda}$ units. Note that temporal snapshots for each model are all taken at the same instant in time throughout this chapter.
Figure 35: Instantaneous radial magnetic fields near the outer shell boundary (top row) and iso-surfaces of axial vorticity (bottom row) for select models. Purple (green) indicates radially outward (inward) directed magnetic fields. Red (blue) indicates cyclonic (anticyclonic) vorticity. Each subplot has its own colorscale. The inner yellow sphere represents the inner core. The outer boundary layer has been excluded for clarity. Below each image is the internal quadrupolarity, $q_I$, and the axial vorticity columnarity, $C_{\omega z}$. 
5.3.2 Flow Characteristics

**Energetics** Figure 21 shows the kinetic energies decomposed into zonal flow, meridional circulation, and non-axisymmetric convection components. The energy is dominantly contained in the convective motions and zonal flows, with less than 1% of the energy derived from meridional circulations. Convective motions contribute about 70% of the kinetic energy for $Pm \geq 2$, and decreases with $Pm$ to about 40% in the non-magnetic case. Relative energy in the meridional circulations is approximately constant across the survey.

![Figure 36: Components of the time-averaged kinetic energies. Here, $ZF$ is zonal flow kinetic energy, $MC$ is meridional circulation kinetic energy, $Conv$ is non-axisymmetric convective kinetic energy. Kinetic energy densities, $\mathcal{E}_k$, are also given.](image)

Magnetic fields also act to reduce both rms and convective flow speeds. The Reynolds numbers ($Re$) and the convective Reynolds numbers ($Re_c$) in models with $Pm \geq 2$
are only about half of those in models with $Pm \leq 1$. These parameters are defined in Table 5 and given in Table 10. Magnetic fields, therefore, promote the relative non-axisymmetric convective component by damping the zonal flows more strongly.

**Convective Flow Structures**  Figure 35 shows snapshots of axial vorticity isosurfaces for the $Pm = 10, 1, \text{ and 0.25}$ models. These flow visualizations illustrate that all of our models are characterized by three-dimensional convection. However, a change in axial vorticity organization occurs as the magnetic Prandtl number is varied. The axial vorticity isosurfaces show that the fluid parcels have randomly distributed vortices with no preferred circulation direction in all models except in the $Pm = 0.25$ model. Here, the region exterior to the tangent cylinder is predominantly composed of anticyclonic vortices while the regions above and below the inner core primarily contain cyclonic vortices.

These visualizations also demonstrate that the length scales of the flow structures differ between the models. Typical length scales are quantified using the characteristic wavenumber of flow, $k_u$, defined in (28) and given in Table 10. These calculations show that the flow structures are nearly a factor of three times smaller-scale in the SMP regime than in either the AMP or AQ regimes.

Convective flow structures are further characterized by axial vorticity columnarity, $C_{\omega z}$, and relative axial helicity, $|H_z^{rel}|$. Recall that columnarity, defined in (29) and calculated using only the non-axisymmetric velocity field, is a measure of the axial
variations of $\omega_z$ in the bulk fluid outside of the tangent cylinder relative to the total unsigned vorticity, $\omega$. We consider cases with $C_{\omega z} \gtrsim 0.5$ to be columnar since quasigeostrophic convection is dominated by axial, vortical columns that extend in $\hat{z}$ across the shell. Relative axial helicity is defined as axial helicity normalized by its maximum possible value in (30). In Chapter 3, we found that columnar flows typically have $|H_{rel}^{z}| \gtrsim 0.1$. These quantities, given in Table 10, show that all models in our survey are characterized by three-dimensional, weakly helical convection since $C_{\omega z} \lesssim 0.4$ and $|H_{rel}^{z}| < 0.1$.

**Zonal Flows and Meridional Circulations** Figure 37 shows the time-averaged zonal winds on the outer shell boundary as a function of latitude and the zonal flows and meridional circulations in the $(r, \theta)$ plane for the $Pm = 10$, 1, and 0.25 models. Two distinct styles of zonal flow are found. A strong retrograde equatorial jet and flanking prograde jets form when $Pm \leq 1$. This regime was discussed in Chapter 4. Zonal flows in models with $Pm \geq 2$ are fundamentally different where, instead, a broad prograde equatorial jet dominates with little motion or weak retrograde flow near and interior to the tangent cylinder. Panel a of Figure 37 indicates that the jet direction sharply flips since no transitional regime was found. Further, the wind speeds in the prograde jet regime are only about half that of those in the retrograde jet regime. Consequently, increasing the electrical conductivity ($Pm$) can cause the direction of zonal jets to reverse.
Figure 37: a) Time-averaged zonal flow profiles on the outer shell boundary. Time-averaged zonal flows and meridional circulations are shown for the b) $Pm = 10$, c) $Pm = 1$, and d) $Pm = 0.25$ cases. Velocity is given in zonal Rossby number units, $Ro_{ZF} = U_{\phi}/\Omega R_O$. Red (blue) indicates prograde (retrograde) zonal flows; clockwise (counterclockwise) poloidal motions are red (blue).
Interestingly, the meridional circulations are rather similar in all of our models. Each hemisphere develops two large circulation cells with opposite polarities within and outside of the tangent cylinder and across the equator. The circulation patterns differ, however, at large cylindrical radii. While smaller cells near the outer shell boundary occur in both regimes, their polarities are reversed. The cells reinforce the large-scale circulation pattern outside the tangent cylinder in the $Pm \leq 1$ models, but oppose it in the $Pm \geq 2$ models.

5.3.3 Heat Flux Patterns

Heat transfer patterns also differ between the behavioral regimes. Figure 38 compares the outer boundary heat fluxes with the time-averaged axisymmetric radial velocity fields and the mean radial convective heat fluxes in the $Pm = 10$, 1, 0.25 models. These comparisons show that thermal emission maxima correlate well with upwelling flows and with positive mean convective heat fluxes, indicating that meridional circulations largely control the convective heat transfer in our models.

All models have enhanced heat flux near the poles because strong upwellings develop here, regardless of the magnetic Prandtl number. A second peak in heat flux of comparable magnitude also occurs in the equatorial regions in the AMP models. This peak is weaker in the AQ and non-magnetic models (cf. Chapter 4). In the high $Pm$ models, radial motions at low latitudes are weak, leading to heat flux minima here.
Figure 38: a) Time-averaged, axisymmetric heat flux profiles on the outer boundary as a function of latitude. Time-averaged, axisymmetric radial velocity fields (left) and mean radial convective heat fluxes (right) for the b) $Pm = 10$, c) $Pm = 1$, and d) $Pm = 0.25$ cases. Radial velocity given in Reynolds number units, $Re = UD/\nu$; red (blue) indicates upwelling (downwelling). Heat flux has been non-dimensionalized by $D/\Delta T$. 
5.4 Regime Transitions

This section discusses the zonal flow and magnetic field morphology transitions.

5.4.1 Zonal Flow Transition

We have shown that increasing the electrical conductivity \((Pm)\) of the fluid causes the direction of zonal jets to reverse. Torques on coaxial cylinders are important towards understanding zonal flow dynamics because they relate to the axial angular momentum balance of the system, \(L_z\). Following Dumberry (2008), the evolution equation for \(L_z\) can be obtained by taking the volume integral of the axisymmetric azimuthal component of the momentum equation, \(F_\phi = F \cdot \hat{\phi}\), multiplied by the cylindrical radius, \(s\):

\[
\frac{\partial L_z}{\partial t} = \hat{z} \cdot \int_V \mathbf{r} \times F dV = -\int_V s F_\phi dV = \int_0^{R_O} s \left( \int F_\phi d\Sigma \right) ds \tag{58}
\]

Here, \(V\) is volume of the fluid shell, \(d\Sigma = s d\phi dz\) is the area over the cylindrical annulus as shown in Figure 39, and \(\int F_\phi d\Sigma\) are the torques on cylindrical annuli.

The dimensional torque balance is obtained by integrating the axisymmetric azimuthal component of the momentum equation (11) over the surface area around the cylinder:

\[
\frac{\partial}{\partial t} \int u_\phi d\Sigma = -\int (u \cdot \nabla u)_\phi d\Sigma + \int \frac{1}{\rho \mu_o} (B \cdot \nabla B)_\phi d\Sigma + \int \nu (\nabla^2 u)_\phi d\Sigma, \tag{59}
\]

The first term on the right hand side is due to advection and Reynolds stresses, the second term represents Taylor torques due to magnetic fields, and the third term is
due to viscous diffusion. Note that there is no pressure term in (59) because

$$\int (\nabla \Pi)_{\phi} d\Sigma = \int_{-z_T}^{z_T} \left( \int_{0}^{2\pi} \frac{\partial \Pi}{\partial \phi} d\phi \right) dz = 0 \quad (60)$$

and no Coriolis term when the fluid is incompressible because

$$\int (2\Omega \times \mathbf{u})_{\phi} d\Sigma = \int 2\Omega \mathbf{u}_s d\Sigma = \int 2\Omega (\nabla \cdot \mathbf{u}) dV = 0. \quad (61)$$

Figure 39: Geometry for zonal torque balance. $d\Sigma$ is the surface area around a coaxial cylinder of radius $s$ and height $2z_T$. The dimensionless torque equation is

$$\underbrace{\frac{\partial}{\partial t} \int u_{\phi} d\Sigma}_{T_t} = \underbrace{-Re \int (\mathbf{u} \cdot \nabla \mathbf{u})_{\phi} d\Sigma}_{T_i} + \underbrace{Q \int (\mathbf{B} \cdot \nabla \mathbf{B})_{\phi} d\Sigma}_{T_L} + \underbrace{\int (\nabla^2 \mathbf{u})_{\phi} d\Sigma}_{T_V} \quad (62)$$

when (59) is non-dimensionalized using dynamically relevant, local length and time scales. Similar to our approach in Chapter 3, we scale velocity as $U$, velocity gradients as $\nabla \mathbf{u} \sim U/\ell_u$, velocity laplacian as $\nabla^2 \mathbf{u} \sim U/\ell^2_u$, magnetic field intensity as $B$, magnetic field gradients as $\nabla \mathbf{B} \sim B/\ell_B$, and time as $\tau_\nu \sim \ell^2_\nu/\nu$ since zonal flows.
evolve slowly. Here, \( \ell_u \) is the length scale over which flow variations occur, \( \ell_B \) is the length scale over which magnetic variations occur, and \( \ell_\nu \) is the length scale at which viscous diffusion occurs. The new dimensionless numbers are a ‘local Reynolds number’, \( Re_\ell \), in which the relative strength of the inertial and viscous forces is estimated by

\[
Re_\ell = \frac{(u \cdot \nabla)u}{\nu \nabla^2 u} \sim \frac{U \ell_u^2}{\nu \ell_u} \sim Re \left( \frac{D}{\ell_u} \right) \left( \frac{\ell_\nu}{D} \right)^2
\]

and a ‘local Chandrasekhar number’, \( Q_\ell \), in which the relative strength of Lorentz and viscous forces is estimated by

\[
Q_\ell = \frac{(\nabla \times B) \times B}{\rho \mu_0 \nu \nabla^2 u} \sim \frac{B^2 \ell_\nu^2}{\rho \mu_0 \nu B U} \sim \frac{\Lambda_d}{E} \left( \frac{\ell_\nu}{D} \right)^2.
\]

Further insight can be gained by rewriting the Taylor torques in the form:

\[
\int (B \cdot \nabla B) \phi d\Sigma = \frac{1}{s^2} \frac{\partial}{\partial s} \left( s^2 \int_{-z_T}^{z_T} \int_0^{2\pi} B_s B_\phi d\phi dz \right).
\]

This assumes that the gradient of the magnetic pressure is zero on coaxial cylinders and that the region exterior to the fluid shell is electrically insulating. If the magnetic field is decomposed into mean and fluctuating components, \( B = \overline{B} + B' \), the Taylor torque becomes

\[
T_L = Q_\ell \int (B \cdot \nabla B) \phi d\Sigma
\]

\[
= \frac{Q_\ell}{s^2} \frac{\partial}{\partial s} \left( s^2 \int_{-z_T}^{z_T} \int_0^{2\pi} B_s B_\phi d\phi dz \right) + \frac{Q_\ell}{s^2} \frac{\partial}{\partial s} \left( s^2 \int_{-z_T}^{z_T} \int_0^{2\pi} B'_s B'_\phi d\phi dz \right).
\]

The first term on the right hand side, \( T_{MT} \), induces torques through magnetic tension due to the mean field; the second term, \( T_{MS} \), induces torques via Maxwell stresses.
due to correlations between fluctuating cylindrical and azimuthal magnetic fields. In our models where the fluctuating field amplitudes exceed the mean magnetic field strength, $T_{MT}$ is expected to be negligible.

Similarly, decomposition of the velocity field into $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ in the inertial term yields

$$T_I = -Re_\ell \int (\mathbf{u} \cdot \nabla \mathbf{u})_\phi d\Sigma$$

$$= -\frac{Re_\ell}{s^2} \frac{\partial}{\partial s} \left( s^2 \int_{-\pi}^{\pi} \int_0^{2\pi} \bar{u}_s \bar{u}_\phi d\phi dz \right) - \frac{Re_\ell}{s^2} \frac{\partial}{\partial s} \left( s^2 \int_{-\pi}^{\pi} \int_0^{2\pi} u_s' u_\phi' d\phi dz \right).$$

Here, $T_{MA}$ induces torques through advection by mean meridional circulations and $T_{RS}$ induces torques via Reynolds stresses due to correlations between fluctuating cylindrical and azimuthal fluid motions. The $T_{MA}$ term is likely important when there is significant cylindrical shear in the zonal flows and/or meridional circulations, while $T_{RS}$ is likely important when the flow field has a substantial non-axisymmetric convective component. Since both of these conditions are satisfied in the models characterized by retrograde equatorial jets, we expect both terms to play a role in the torque balance. In contrast, we expect $T_{RS} > T_{MA}$ in the models characterized by prograde equatorial jets because the cylindrical shear is weaker and more of the kinetic energy is contained in convective motions than in axisymmetric flows.

We consider two cases on either side of the zonal flow transition for simplicity: $Pm = 1$ which has a retrograde equatorial jet and $Pm = 2$ which has a prograde equatorial jet. Figure 40 shows the torque balance as a function of cylindrical radius outside of the
tangent cylinder at a snapshot in time for each model. These profiles indicate that the
torque balance in both models fluctuates significantly in time since the temporal term,
$T_t$, is dominant. These variations are balanced by the fluctuating inertial term due to
Reynolds stresses. No significant difference between the instantaneous torque balances
are apparent, so we also consider the time-averaged torque balance in Figure 41.

The time-averaged torque balance is obtained by calculating each time-averaged term
in (62). However, since the forces were not output during the production runs, the
calculations were made using the time-averaged velocity and magnetic fields. This
approach then neglects contributions due to fluctuating fluid motions ($T_{RS}$) and mag-
netic fields ($T_{MS}$). These contributions can be indirectly determined by averaging over
times long enough such that $T_t \approx 0$ and taking the residual:

$$T' = T_{RS} + T_{MS} = - (\overline{T_{MA}} + \overline{T_{MT}} + \overline{T_V}),$$

where overbars indicate calculations made using time-averaged $u$ and $B$. Each term
on the right hand side of (68) has been time-averaged over at least 0.2 diffusion times.
Temporal variations are secondary in both models because the zonal flows outside of
the tangent cylinder are approximately constant in time. The instantaneous torque
calculations suggest that the residual term is primarily due to Reynolds stresses, $T_{RS}$,
in both models. However, Maxwell stresses may also play a role in the $Pm = 2$ model
since $T_{MS}$ is not negligible at large cylindrical radii.
Figure 40: Snapshot of the torque balance as a function of radius outside of the tangent cylinder following (62) in the a) $Pm = 1$ and b) $Pm = 2$ models. Positive (negative) torques produce prograde (retrograde) zonal flows.
Figure 41: Time-averaged torque balance as a function of radius outside of the tangent cylinder following (62) in the a) \( Pm = 1 \) and b) \( Pm = 2 \) models. Each term has been time-averaged over at least 0.2 viscous diffusion times. Positive (negative) torques produce prograde (retrograde) zonal flows.
Figure 41 shows that there is an approximate balance over time between the inertial and viscous diffusion terms ($T_{MA} + T_{RS} \approx T_V$) in both models. Interestingly the mean advection term, $T_{MA}$, behaves similarly across the transition where the fluid is torqued in the retrograde direction near the tangent cylinder. This occurs because the meridional circulations are strongest here and the circulation patterns are similar in each model. The key difference between these models is the direction in which the fluctuating stresses torque the fluid. Since Reynolds stresses are presumed to be dominant in both models based on the instantaneous torques, fluctuating fluid motions transfer angular momentum in opposite directions in these two cases. We do note, however, that Maxwell stresses may also be important in the $Pm = 2$ model, particularly away from the tangent cylinder. Maxwell stresses can then alter the zonal flow directly or indirectly by modifying the Reynolds stresses (e.g., Browning, 2008). Both of these effects may be important in the high $Pm$ models, although the torque snapshots suggest that the latter may be more prevalent.

These calculations also allow our predictions about the relative contributions of the mean and fluctuating components of the inertial and Taylor torques to be tested. For example, Taylor torques due to time-averaged axisymmetric magnetic fields are relatively weak, in agreement with our prediction since the fluctuating field amplitudes exceed the mean magnetic field strength. Similarly, our prediction that $T_{RS} \approx T_{MA}$ in the $Pm = 1$ model and $T_{RS} > T_{MA}$ in the $Pm = 2$ model also holds over time-averages.
5.4.2 Magnetic Field Transitions

This section investigates the magnetic field morphology transitions between strongly multipolar, axial multipolar, and axial quadrupolar dynamos. Figure 42a plots the ratio of temporally- and spatially- averaged signed to unsigned magnetic field strengths in our models as a function of the magnetic Prandtl number. This comparison indicates that dynamo action is occurring on both local and global scales (Cattaneo, 1999). In our high magnetic Prandtl number models, the local dynamo exceeds the mean field dynamo by almost an order of magnitude such that magnetic fields are primarily driven by fluctuating dynamo action due to the random stretching and amplification of magnetic fields via convective fluid motions. In contrast, the local and global magnetic fields are comparable in our two lowest $Pm$ models such that both fluctuating and mean field dynamo action are important.

The mode of magnetic field generation can be further examined by considering the behavior of the dipole component. Figure 43 shows the latitude of the dipole, $\theta_{dip}$, as a function of time for models with $Pm = [0.25, 1, 10]$. A significant trend in the reversal frequency is observed. One reversal occurs within three viscous diffusion times in the $Pm = 0.25$ model, and this rate increases with $Pm$. However, it is difficult to define what constitutes a reversal in the higher $Pm$ models where the pole location randomly fluctuates. Instead, this trend is also reflected in the mean unsigned latitude of the dipole. Since the dipole typically prefers to be oriented such that it is nearly aligned with the rotation axis, the mean unsigned latitude of the dipole is expected to be
near 90° when the dipole is stable. In contrast, \( \theta_{dip} \) is expected to approach 45° as the pole fluctuates randomly in latitude.

Figure 42b shows the mean time-averaged dipole latitudes in our models. The average dipole latitude decreases with increased magnetic Prandtl number from about 60° in the \( Pm = 0.25 \) model to about 40° in the \( Pm = 10 \) model. This trend is consistent with the mean field becoming less prevalent compared to the fluctuating field as \( Pm \) is increased. In addition, we further note that \( \theta_{dip} < 50° \) in the \( Pm \geq 2 \) models; this result explains why the equatorial \( m = 1 \) spherical harmonic order exceeds the axisymmetric \( m = 0 \) mode in the SMP regime.

In contrast, the reversal patterns suggest that the magnetic fields increasingly manifest as dynamo waves as the magnetic Prandtl number is decreased. Here, the magnetic fields are passively carried by the zonal flows and meridional circulations. It is then not surprising that the dipole and quadrupole magnetic field components are prominent in our low \( Pm \) oscillatory dynamos (cf. Grote et al., 1999, 2000a; Simitev and Busse, 2005; Busse and Simitev, 2006). It is not presently understood why a quadrupolar, rather than dipolar, solution is preferred in our \( Pm = 0.25 \) model. We note, however, that quadrupolar dynamos are found only near the onset of dynamo action in both our inertially-dominated models and in the rotationally-dominated models of the literature (Grote et al., 1999, 2000a; Simitev and Busse, 2005; Busse and Simitev, 2006).
Figure 42:  a) Ratio of temporally- and spatially- averaged signed to unsigned magnetic field strengths and b) time-averaged unsigned latitude of the dipole as a function of the magnetic Prandtl number. The error bars indicate one standard deviation.
Figure 43: Time series of the dipole latitude for cases representative of each regime. The transient behavior has been excluded, and the time series is chosen to begin at zero for simplicity.
5.5 Discussion

It is often assumed that the dynamics in planetary dynamo models are robust. However, these models are computationally limited to Ekman numbers that greatly exceed planetary estimates, such that viscosity is large compared to that expected in planetary settings. The magnetic Prandtl number must then also be artificially large in order for dynamo action to occur. Despite this disparity in parameter values, model results are often compared against planetary observations under the premise that no fundamental changes occur in the flow or magnetic fields as model results are extrapolated to planetary settings since both systems tend to have similar levels of magnetic induction ($Rm$) (e.g., Christensen and Wicht, 2007). This assumption implies that simulations have reached the asymptotic regime in $Pm$ space. This hypothesis is tested using strongly-forced simulations which vary only the magnetic Prandtl number, and we find that the fluid dynamical, heat transfer, and dynamo characteristics all fundamentally change with $Pm$ such that the asymptotic $Pm$ behavior has not been attained in our models.

The magnetic Reynolds number, however, varies by more than an order of magnitude across the survey: $300 \lesssim Rm \lesssim 5000$. As such, the simulations presented here alone are not sufficient to test if the magnetic Reynolds number controls the applicability of model results to planetary settings. Towards this end, I am currently running additional models with the magnetic Reynolds number fixed to $Rm \sim 10^3$ and $Pm = [0.25, 10]$ to supplement this survey.
This work does not invalidate the ice giant-style dynamo models of Chapter 4. The zonal flow pattern appears to be robust as $Pm$ is decreased, as does the peak thermal emissions in the polar regions. It is difficult to determine, however, whether the equatorial emission peak will develop at the low $Pm$ values in planetary dynamo regions. Magnetic power spectra of the ice giants indicate that the power contained in the dipole, quadrupole, and octupole components is comparable to within a factor of two (Figure 30). Our results are consistent with these observations, given their sizeable uncertainty, since there is never more than a factor of five difference between the $l \leq 3$ spherical harmonic degrees in our models (Figure 33).

Our study is also complementary to the work of Simitev and Busse (2005) who carry out a similar suite of planetary dynamo simulations in the rotationally-dominated regime. A comparison of our results shows that dynamos tend to be controlled by the local convective flows at relatively high magnetic Prandtl numbers, regardless of the convective regime. Consequently, for rapidly-rotating convection, local axially-aligned columns tend to generate stable axial dipoles. In contrast, three-dimensional turbulence that occurs in the inertial regime instead tends to generate poorly-organized, small-scale dynamos.

A remarkably different behavior is observed at relatively low magnetic Prandtl numbers. Here, despite having dramatically different convective flows, similar magnetic field morphologies are obtained in our inertial models and in the rapidly-rotating models of Simitev and Busse (2005). This result suggests that the dynamos are driven
differently here compared to the high $Pm$ models. Strong zonal flows develop in both sets of low $Pm$ models because the mechanical boundary conditions are stress-free. Further, these quasi-steady zonal flows with high Reynolds numbers will have low magnetic Reynolds numbers for small $Pm$ values. We, therefore, do not expect magnetic fields to back react strongly on the flows. This scenario is essentially a kinematic dynamo since the mean flows are quasi-steady and approximately independent of the magnetic field where the field is generated by a macroscopic $\alpha\Omega$-effect (cf. Roberts, 1972; Dudley and James, 1989; Arlt, 2006). It is not presently understood, however, when and why a quadrupolar, rather than a dipolar, solution will be preferred. These results lead us to hypothesize that large-scale flows control dynamo action when both $Rm$ and $Pm$ are low, implying that the dynamo is less sensitive to small-scale flows and, therefore, to the convective regime.
6 Concluding Remarks

In this thesis we have carried out two complementary surveys of terrestrial-style and strongly-forced numerical simulations to investigate behavioral transitions in planetary dynamo models. These surveys allow us to explore the coupling between fluid motions, heat transfer, and magnetic field generation across parameter space. Our results are briefly summarized below; I also recommend directions for future work.

Chapter 3 examines the dynamics in planetary dynamo and non-magnetic rotating convection models with no-slip boundary conditions, thought to be appropriate for terrestrial bodies. These simulations allow us to quantify how the dynamics change as the thermal forcing is increased with all other control parameters fixed and to assess how magnetic fields influence convection. We find two heat transfer regimes and show that scaling laws from non-magnetic planar convection also apply to our models, despite the spherical geometry and regardless of magnetic field strength. This regime transition occurs when the thermal and Ekman boundary layer thicknesses are comparable and coincides with the breakdown of columnar convection into poorly-organized, three-dimensional motions. We further demonstrate that the convective (non-zonal) flow structures and speeds are also not significantly affected by the presence of magnetic fields.

The traditional measure of magnetic field strength, $\Lambda$, however, predicts that the Lorentz force should have a first order contribution to the force balance, and, there-
fore, to play an important dynamical role in the majority of our models. We argue that this parameter is only applicable for systems with quasi-static magnetic fields. This restriction is likely invalid in dynamo models and in planetary interiors, which explains why it overestimates the influence of the Lorentz force. Instead, we show that a dynamic Elsasser number, $\Lambda_d$, accurately measures the Lorentz to Coriolis force ratio. Extrapolating our results to planetary interiors, we predict that the Lorentz force due to observable, large-scale magnetic fields should be weaker than the Coriolis force in all of the planets.

A sharp transition between dipolar and multipolar dynamos is also found as the thermal forcing is increased. This breakdown of the dipole coincides with the degradation of helical flow and occurs when the ratio of inertial to viscous forces is order unity. We, therefore, hypothesize that large-scale magnetic field generation occurs in the majority of present day dynamo models (e.g., $E \gtrsim 10^{-4}$ and $Pm \sim Pr \sim 1$) via viscously-stabilized helical flow.

These results highlight the need for dynamo simulations to be carried out at lower Ekman and magnetic Prandtl numbers. Through such an extended dataset, we may be able to empirically determine a scaling law for the dynamic Elsasser number, $\Lambda_d$, in terms of the model control parameters. Presently, $\Lambda_d$ depends on three output parameters: $\Lambda_i$, $Rm$, and $\ell_B$. This direction for future work is necessary in order to determine how $\Lambda_d$ scales to low Ekman and magnetic Prandtl numbers appropriate for planetary interiors. We are currently limited to estimating the traditional Elsasser
number, $\Lambda_i$, and the magnetic Reynolds number, $Rm$, from observations and assuming that magnetic fields vary on length scales comparable to the thickness of the dynamo region, $\ell_B \sim D$. This approach for predicting the dynamic Elsasser number yields a lower bound estimate; the influence of smaller-scale magnetic fields on convective (non-zonal) dynamics may be more pronounced.

Future work should also strive to determine how the transition between dipolar and multipolar dynamos scales with the Ekman and magnetic Prandtl numbers and to test the limits of dipolar magnetic field generation via viscously-stabilizes helices. In other words, lower Ekman number models are necessary in order to formulate a transition criteria that is not linked to the viscous force. Dynamo models with stress-free boundaries may also be used towards this end (Simitev and Busse, 2005; Busse and Simitev, 2006, 2011).

The dichotomy between dipole-dominated gas giants and multipolar ice giants is investigated in Chapter 4. Here, we test the hypothesis that strong inertial effects control the generation of ice giant-style dynamics. Towards this end, we present strongly-forced dynamo and non-magnetic models and show that inertially-dominated convection generates zonal flow, heat flux, and magnetic field patterns that qualitatively agree with observations of the ice giants.

In these models, angular momentum mixing transports angular momentum inward to drive a retrograde jet at large cylindrical radii and prograde flow closer to the rotation axis. In addition, the heat flux pattern is coupled to the meridional circu-
lations, and the magnetic fields are driven by fluctuating dynamo action where the poorly-organized, three-dimensional convective motions randomly stretch and amplify magnetic fields.

Two directions for future work are evident in this study. First, it is important to understand what controls the transition to inertial convection and to determine how this transition scales to planetary settings. This is an active area of research (e.g., Sprague et al., 2006; Schmitz and Tilgner, 2009, 2010; King et al., 2009, 2010). The most precise convective regime transition scalings to date are from King et al. (2009, 2010). These studies suggest that convective regimes may be controlled by boundary layer physics, particularly by the relative thicknesses of the thermal and Ekman boundary layers. However, uncertainties in the scaling law and in the physical properties of the planets (e.g., diffusivities) presently preclude definitive predictions of the planetary convective regimes.

Second, the next step to make the models presented in Chapter 4 more planet-like is to incorporate radially-varying density and electrical conductivity (e.g., Heimpel and Gomez-Perez, 2011). This allows us to simulate both the electrically insulating molecular envelope and the electrically conducting ionic ocean in a single, self-consistent model. It is, therefore, important to carry out strongly-forced anelastic dynamo models. Our non-magnetic and dynamo Boussinesq models then serve as end-member cases for such an anelastic model.

Inertially-dominated systems tend to have large Reynolds numbers with $Re \geq O(10^3)$. 
Consequently, even models with low magnetic Prandtl numbers can have magnetic Reynolds numbers that exceed the critical value: $Rm = RePm \gtrsim 10^2$. We then also systematically quantify the dynamical consequences of varying the magnetic diffusivity on the magnetic and flow fields over a relatively large range of $Pm$ values, covering almost two orders of magnitude, which had not been previously examined in this convective regime.

This suite of simulations allows us to test the common assumption that no fundamental changes occur in the flow or magnetic fields as model results are extrapolated to low magnetic Prandtl numbers since the simulated $Rm$ values are similar to planetary estimates (e.g., Christensen and Wicht, 2007). We find that the fluid dynamical, heat transfer, and dynamo characteristics all change with $Pm$. In particular, the magnetic field is shown to transition from a dynamo wave solution near the onset of dynamo action to a poorly-organized small-scale dynamo as the magnetic Prandtl number is increased. Furthermore, the flow direction of the equatorial zonal jet is found to reverse with increased $Pm$. The outer boundary thermal emission pattern is also affected by changes in the magnetic Prandtl number. These results imply that asymptotic $Pm$ behavior has not been attained in our models.

These dynamical changes that result from varying the electrical conductivity of the system ($Pm$), however, also cause the magnetic Reynolds numbers to differ by more than an order of magnitude across our survey. This variation is inconsistent with the hypothesis that we are testing since it assumes $Rm$ to be approximately constant.
The next step of this project is then to fix the magnetic Reynolds number and instead vary the magnetic Prandtl number.

Our strongly-forced survey can be used, however, to make some general inferences about the nature of magnetic field generation through a comparison against a complementary survey carried out in the rotationally-dominated regime by Simitev and Busse (2005). First, dynamos tend to be controlled by the local convective flows at relatively high magnetic Prandtl numbers, regardless of the convective regime. In addition, large-scale flows control dynamo action when both $Rm$ and $Pm$ are relatively small, implying that the dynamo is less sensitive to small-scale flows and, therefore, to the convective regime.

Dynamo models yield a diverse spectrum of dynamical behaviors and are able to reproduce aspects of the planets’ magnetic fields, fluid flows, and thermal emissions. Thus, through these models, we learn about the dynamics and underlying mechanisms that may be occurring in planetary interiors. Technological advances will further our understanding of these systems by enabling models to reach lower Ekman and magnetic Prandtl numbers, to develop asymptotic scaling laws, and to incorporate significant stratification in density and electrical conductivity. By understanding the dynamos of our solar system, we may also use these models and associated scaling laws to predict the magnetic field strengths and morphologies of exoplanetary and stellar dynamos.

Spacecraft missions will also provide additional constraints on the magnetic fields,
dynamics, and internal structures. For example, the MESSENGER mission will help constrain the behavior of Mercury’s magnetic field and the structure of the planet’s dynamo region (Anderson et al., 2011). The upcoming Juno mission will critically improve our understanding of Jupiter by striving to resolve the magnetic field up to spherical harmonic degree 14, to detect short timescale secular variation, and to constrain the depth of the zonal winds. The Cassini spacecraft may closely orbit, and eventually impact, Saturn in the possible extended-extended phase of the mission, allowing the magnetic and gravitational fields to be mapped in further detail (*Cassini Mission Plan, Revision N*). In addition, a possible future mission to the Jupiter system would help to characterize Ganymede’s dynamo and internal structure. Crucially, missions to Uranus and/or Neptune are necessary to constrain the internal structures and dynamics of the ice giants and to map the magnetic fields globally and with higher resolution. This data will hopefully lead to realistic models that can explain the observational dichotomy between the gas and ice giants.

I am optimistic and excited about the future numerical capabilities and observational discoveries that will progress the study of planetary magnetic fields.
References


for quasi-geostrophic rapid dynamics within the Earth’s outer core. *Phys. Earth Planet. Int.*


Iess, L., Rappaport, N., Jacobson, R., Racioppa, P., Stevenson, D., Tortora, P.,


of Mercury’s magnetosphere with applications to the MESSENGER mission and


Karkoschka, E. (2011). Neptune’s rotational period suggested by the extraordinary


Khan, A., Mosegaard, K., Williams, J., and Lognonné, P. (2004). Does the Moon
possess a molten core? Probing the deep lunar interior using results from LLR and

Evidence of a global magma ocean in Io’s interior. *Science*.

King, E. and Aurnou, J. (2011). Thermal evidence for taylor columns in turbulent,

King, E., Soderlund, K., Christensen, U., Wicht, J., and Aurnou, J. (2010). Convec-
tive heat transfer in planetary dynamo models. *Geochemistry, Geophysics, Geosystems*, 11:Q06016.


Malavergne, V., Toplis, M., Berthet, S., and Jones, J. (2010). Highly reducing con-


Smith, B., Soderblom, L., Banfield, D., Barnet, C., Beebe, R., Bazilevskii, A.,
Bollinger, K., Boyce, J., Briggs, G., and Brahic, A. (1989). Voyager 2 at Nep-

Smith, B., Soderblom, L., Beebe, R., Bliss, D., Brown, R., Collins, S., Boyce, J.,

Smith, D., Zuber, M., Phillips, R., Solomon, S., Neumann, G., Lemoine, F., Peale,
S., Margot, J.-L., Torrence, M., Talpe, M., Head, J., Hauck, S.A., L., Johnson,
C., Perry, M., Barnouin, O., McNutt, R., and Oberst, J. (2010). The equatorial
shape and gravity field of Mercury from MESSENGER flybys 1 and 2. *Science*,
209:88–100.

Smith, E., Davis, L., Jones, D., Coleman, Jr., P., Colburn, D., Pyal, P., and Sonett,
C. (1975). Jupiter’s magnetic field, magnetosphere, and interaction with the solar

Smith, E., Davis, L., Jones, D., Coleman, Jr., P., Colburn, D., Pyal, P., Sonett,
C., and Frandsen, A. (1974). The planetary magnetic field and magnetosphere of


