An Investigation of Planetary Convection:  
The Role of Boundary Layers

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Geophysics and Space Physics

by

Eric M. King

2009
The dissertation of Eric M. King is approved.

_____________________
James C. McWilliams

_____________________
Paul H. Roberts

_____________________
Christopher T. Russell

_____________________
Gerald Schubert

_____________________
Jonathan M. Aurnou, Committee Chair

University of California, Los Angeles

2009
For my parents
Contents

1 Introduction 1

1.1 Geophysical Context 2

1.2 Simulating Planetary Core Convection 10

1.2.1 Rotating Magnetoconvection 10

1.2.2 Governing Equations 12

1.2.3 Scaling Analysis 16

2 The Rotating Magnetoconvection Device: Experimental Anatomy 24

2.1 RoMag: an Overview 26

2.2 Spirit of RoMag 28

2.2.1 The Magnet 29

2.2.2 Other Design Considerations 31

2.3 Musculoskeletal System 33

2.4 Circulatory System 39

2.5 Gastrointestinal System 46

2.5.1 Fluid Properties 46

2.5.2 The Tank 48

2.5.3 Storage and Filling 55

2.6 Nervous System 60

2.6.1 Experimental Control 61
2.6.2 Thermometry ................................................. 61
2.6.3 Doppler Velocimetry .......................... 68
2.6.4 Flow Visualizations ................................. 75
2.7 Sequence of Operations ................................. 77
2.8 Experimental Capabilities ............................. 84
2.9 Laboratory-Numerical Collaboration ............... 89

3 Plane Layer Rotating Convection .................. 91
3.1 Heat Transfer Background ............................... 92
3.2 Heat Transfer Results: The Transition ............ 94
3.3 The Role of Boundary Layers ......................... 101
3.4 Exploring the Transition: the breakdown of quasigeostrophic structures 109
3.5 A Stress Free Transition ............................... 121
3.6 Conclusions and Further Considerations .......... 125

4 Dynamo Model Analysis ................................. 128
4.1 Heat Transfer in Spherical Dynamo Models ....... 129
4.1.1 Introduction ............................................ 129
4.1.2 Numerical Model Setup ............................ 132
4.1.3 Heat Transfer Results ............................... 135
4.1.4 Heat Transfer Scaling Regimes .................... 139
4.2 Implications of Boundary Layer Transitions in Dynamo Models ....... 144
| 5 | Radial profiles of experimental magnetic field strength within the solenoid’s hollow bore. | 31 |
| 6 | Experimental magnetic field strength versus amperage. | 32 |
| 7 | RoMag’s lower frame. | 34 |
| 8 | An example of temporal fluctuations in experimental rotation rate. | 36 |
| 9 | A photograph of RoMag’s frame assembly. | 37 |
| 10 | The two positions of RoMag’s electromagnet. | 38 |
| 11 | Magnet lift assembly. | 39 |
| 12 | A schematic of RoMag’s circulatory system. | 40 |
| 13 | An illustration of RoMag’s cooling block. | 42 |
| 14 | A schematic of RoMag’s experimental thermal circulation system. | 44 |
| 15 | A schematic of RoMag’s convection tank stack. | 45 |
| 16 | A design drawing of the bottom thermal block. | 52 |
| 17 | A schematic of RoMag’s nervous system. | 59 |
| 18 | An example Proteus totalizer test. | 65 |
| 19 | Proteus flowmeter calibration curve fit. | 66 |
| 20 | A design drawing of the top thermal block. | 67 |
| 21 | Seed particles in suspension in sucrose solution. | 70 |
| 22 | Density mismatch ratio versus seed particle settle time. | 71 |
| 23 | A design drawing for sidewalls with doppler probe holders | 72 |
| 24 | Planform of doppler chord profile geometry. | 74 |
25 Raw spin-up velocity data from an instantaneous doppler chord profile measurement 75
26 Raw spin-up velocity time series in sucrose solution .................. 76
27 Sample temperature time series measurements ....................... 81
28 Experimental accessibility of Rayleigh-space in liquid metal .......... 85
29 Experimental accessibility of Rayleigh-Ekman-space in water .......... 86
30 Experimental accessibility of Rayleigh-Ekman-space in gallium ...... 87
31 Experimental accessibility of Rayleigh-Chandrasekhar-space in gallium .... 88
32 Experimental accessibility of Ekman-Chandrasekhar-space in gallium ... 89
33 Nusselt number, $Nu$, versus Rayleigh number, $Ra$ .................. 95
34 The transition from rotationally controlled to non-rotating heat transfer behavior. 97
35 Nusselt number versus the convective Rossby number ............... 99
36 Convective regime transitions from previous studies ................. 100
37 Vertical profiles of a) mean temperature, b) temperature variance, and c) RMS velocity .......................... 103
38 Thermal boundary layer thickness measurements ..................... 105
39 The non-dimensional thicknesses of the competing boundary layers .... 106
40 Boundary layer control of heat transfer transitions .................. 107
41 Visualizations of flow from the two regimes ......................... 110
42 Temperature correlations from vertical thermistor pairs in subtransitional rotating convection .................. 111
43 Temperature correlations from vertical thermistor pairs in non-rotating convection. 112
<table>
<thead>
<tr>
<th>Page</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>44</td>
<td>Vertical temperature cross-covariance versus time lag from a one-dimensional heat transfer model</td>
</tr>
<tr>
<td>45</td>
<td>Vertical temperature correlation coefficient as a function of $Ra/Ra_t$</td>
</tr>
<tr>
<td>46</td>
<td>Normalized temperature gradients at the mid-plane versus the boundary layer thickness ratio</td>
</tr>
<tr>
<td>47</td>
<td>Temporal temperature variance at the tank’s center versus $Ra/Ra_t$</td>
</tr>
<tr>
<td>48</td>
<td>Iso-surfaces of vertical velocity from numerical simulations with stress-free boundary conditions.</td>
</tr>
<tr>
<td>49</td>
<td>The Nusselt number as a function of the Rayleigh number for convection with free-slip boundaries.</td>
</tr>
<tr>
<td>50</td>
<td>Heat transfer scaling transition in rotating convection with free-slip boundaries.</td>
</tr>
<tr>
<td>51</td>
<td>The theoretical Ekman-Hartmann layer thickness versus the Elsasser number</td>
</tr>
<tr>
<td>52</td>
<td>A schematic illustration of the dynamo model geometry</td>
</tr>
<tr>
<td>53</td>
<td>Heat transfer behavior, $Nu$ versus $Ra$, from a broad array of convective dynamo models</td>
</tr>
<tr>
<td>54</td>
<td>Heat transfer transitions from the broad array of convective dynamos</td>
</tr>
<tr>
<td>55</td>
<td>Boundary layer thicknesses versus $Ra/Ra_t$</td>
</tr>
<tr>
<td>56</td>
<td>Bounding the range of applicability of the diffusivity free scaling law.</td>
</tr>
<tr>
<td>57</td>
<td>Meridional slices of time-averaged temperatures from dynamo models</td>
</tr>
<tr>
<td>58</td>
<td>Mean radial temperature profiles from dynamo models</td>
</tr>
<tr>
<td>59</td>
<td>Temperature lapse rate at mid-shell versus $Ra/Ra_t$</td>
</tr>
</tbody>
</table>
List of Tables

1 Nondimensional parameters used to characterize planetary fluid behavior . . . . . . 7
2 Thermophysical properties of water, sucrose solution, and gallium . . . . . . . . 27
3 Typical nondimensional parameters used in dynamo modeling . . . . . . . . . . 134
ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Jon Aurnou, for his patient and dedicated support through all my strikes and gutters. He selflessly afforded me his ideas and enthusiasm, yet encouraged me to learn independently to develop ideas of my own. Thanks are also due to the rest of my doctoral committee, Jim McWilliams, Paul Roberts, Chris Russell, and Jerry Schubert, for their critical feedback and ardent counsel, which significantly improved this document.

Further thanks are due to Jerome Noir for his guidance and friendship, especially early on. To Afshin Yaghmaei, for the many days we spent carefully troubleshooting the experiment. To Stephan Stellmach, without whose formidable numerics and stalwart analysis this work would not have been possible. To Krista Soderlund, who worked diligently to provide dynamo model data, and was patient through an at-times circuitous collaboration. To Uli Christensen (and his collaborators), for providing me with his carefully shepherded flock of dynamos. To fellow graduate students Mr. Michael Calkins and Bill P. Burgess, for gainful discussions and distractions. And to all others I have not mentioned, professional and personal, who have lent me their experiences so that I may learn.

This work was supported by the Earth Sciences Division Geophysics Program and Instrumentation and Facilities Program of the United States National Science Foundation, as well as by the Planetary Atmospheres Program of the National Aeronautics and Space Administration.
Vita

Born, Albany, New York, 1982

B.Sc. Mathematical Physics, Binghamton University, 2004, *Magna Cum Laude*

M.Sc. Geophysics and Space Physics, University of California, Los Angeles, 2007

Graduate Student Researcher, University of California, Los Angeles, 2004–2009

Graduate Fellow, University of California, Los Angeles, 2005, 2007, 2008

Teaching Assistant, University of California, Los Angeles, 2006–2009

Publications and Presentations


Abstract of the Dissertation

An Investigation of Planetary Convection:

The Role of Boundary Layers

by

Eric M. King

Doctor of Philosophy in Geophysics and Space Physics
University of California, Los Angeles
Professor Jonathan Aurnou, Chair

Thermal and gravitational energy sources drive turbulent convection in Earth’s vast liquid metal outer core. These fluid motions generate the electric currents that are believed to power Earth’s magnetic field through a process known as dynamo action. Core flow is subject to the influence of Earth’s rotation via the Coriolis force, which has an organizational effect on otherwise chaotic motions. Furthermore, the magnetic field generated by convection acts back on the flow via Lorentz forces. Fluid motions in Earth’s core, and the magnetic field generating regions of other planets and stars, are then governed by three main ingredients: convection, rotation, and magnetic fields. The goal of my Ph.D. research is to further our understanding of the systematic fluid dynamics occurring in dynamo systems.
To accomplish this, I have developed a unique experimental device that allows me to produce fluid conditions approaching those expected in Earth’s core and other planetary and stellar environments. The results presented here stem from a broad parameter survey of non-magnetic, rotating convection. In this study, I examine the interplay between rotation and convection by broadly varying the strength of each and measuring the efficiency of convective heat transfer. This parameter survey allows me to argue that the importance of rotation in convection dynamics is determined by boundary layer physics, where the Ekman (rotating) and thermal (non-rotating) boundary layers compete for control of convection dynamics. I develop a simple predictive scaling of this convective regime transition using theoretical boundary layer thickness scalings. This transition scaling permits a unified description of heat transfer in rotating convection, which reconciles contrasting results from previous studies. I also extend this experimental result to a broad array of numerical dynamo models, arguing that the boundary layer control of convective regimes is also evident in the dynamo models. A notable implication of this regime transition scaling is that it is much easier than previously thought for non-rotating style dynamics to occur in convection experiments and simulations, and perhaps also in planets and stars.
1 Introduction
1.1 Geophysical Context

The magnetic fields captured in ancient crustal rocks tell us that Earth has had a magnetic field for at least 3.5 billion years (e.g., Hale and Dunlop (1984)). This magnetic field shields us from the energized particles blasted earthward by the Sun (Kivelson and Russell (1995)). The solar bombardment sneaks in through polar kinks in this magnetic armor to produce the aurora borealis and aurora australis, or the northern and southern lights. This phenomenon reveals the general shape of Earth’s magnetic field: similar to that of a bar magnet, with magnetic field lines leaving the south pole, circling around and reentering near the north pole. We know, though, that Earth’s interior is too hot to maintain a giant internal bar magnet, as minerals above their Curie temperature (≈ 600K) cannot be permanently magnetized (Stacey (1969)). So where does our magnetic field come from?

Earth’s deep interior is comprised of three main regions (figure 1): at the center, a solid iron inner core; above that the liquid metal outer core, a vast ocean of iron (78-88%), nickel (∼5%), and some lighter element(s) (5-10%) (McDonough and Sun (1995)); and the outermost ‘solid’ mantle. The electrically conducting inner half of the planet provides a good candidate for the source of Earth’s magnetic field. Perhaps the geomagnetic field is a relic from planetary formation? This is highly unlikely, since any magnetic field locked into the primordial Earth would have long since decayed away. To see this, we can estimate a conservative magnetic diffusion time for the Earth:

\[ \tau_B = \frac{L^2}{\pi^2 \eta} \]

where \( L \) is the size of the conducting region, and \( \eta \) is its magnetic
Figure 1: A schematic depiction of Earth’s gross interior structure. Three basic layers are shown that are, from inner to outer: the inner core, the outer core, and the mantle. The boundary between the solid inner core and liquid outer core is the inner core boundary (ICB), which has an average radius of 1221 km. The boundary between the outer core and mantle, called the core-mantle boundary (CMB), has a radius of about 3480 km (Lowrie (1997)). The mantle, shown here to be uniform and stagnant, does, in fact, host dynamics of critical importance to global geophysics (Turcotte and Schubert (2002)).
diffusivity (Davidson (2001)). Earth’s core, with its 3,480 km radius, composed of a provisionally magnetized metal \( (\eta \approx 1 \text{ m}^2/\text{s}) \), would see its magnetic field decay in about 40,000 years. Observations tell us, however, that Earth’s magnetic field has been around approximately 100,000 times as long.

The longevity of Earth’s magnetic field indicates that it must be continuously generated against its tendency to decay. It is now widely thought that the geomagnetic field is generated by electrical currents produced by motion in the liquid metal outer core. The conversion of kinetic energy to electromagnetic energy is known as dynamo action. Researchers indirectly observe that the field has undergone sporadic polarity reversals over the past hundred million years (Johnson et al. (2003)), and directly observe more subtle variability on much shorter timescales (Bloxham et al. (1989)), which support the idea of dynamic magnetic field generation through fluid motions.

The conversion of kinetic energy to magnetic energy requires an energy source for fluid motions. There are many different ways to drive flow in the core. For example, irregularities in the rotation of Earth’s mantle such as differential rotation, precession and libration, driven, for example, by tidal dissipation by way of orbital dynamics, can generate fluid motions via core-mantle coupling (e.g., Dormy et al. (1998), Malkus (1968), Noir et al. (2009)).

Perhaps the most likely energy source for dynamo action, however, comes from convective instability. Earth’s core has been cooling and differentiating since its formation, and this redistribution of heat and mass is capable of driving vigorous fluid
motions. Heat left over from planetary formation and heat from the radioactive decay of alloying core material such as potassium ($^{40}$K) (Nimmo et al. (2004)) are removed from the core by mantle convection, cooling and therefore destabilizing fluid near the core-mantle boundary (CMB). The growing inner core contributes a buoyancy flux at the inner core boundary (ICB), as crystalizing iron deposits latent heat and anomalous concentrations of lighter elements at the bottom of the liquid core (e.g., Stacey (1969)). As W. M. Elsasser once noted, “there are too many ways in which the core can be made to convect to permit an unambiguous interpretation” (Davidson (2001)).

Observations of Earth’s magnetic field provide valuable insight into core dynamics. First, we notice that Earth’s field is largely dipolar, and the dipolar component of the magnetic field is nearly aligned with Earth’s rotation axis. This symmetry hints at the importance of Earth’s rotation on convection dynamics. Observations of the aforementioned temporal variations in Earth’s field allow us to probe indirectly convective fluid motions in the core, as flowing liquid metal sweeps along with it magnetic field features. Assuming perfect advection of field lines (which is not a bad assumption on timescales longer than about a decade), researchers estimate a roughly 0.1 mm/s westward flow at the core-mantle boundary (CMB) (Bloxham and Jackson (1991)).

While 0.1 mm/s may seem slow, the size of the core implies that this a turbulent body of fluid. The Reynolds number characterizes the vigor of fluid flow, $Re = UL/\nu$,
where $U$ is the typical velocity, $L$ a typical length scale, and $\nu$ is the fluid’s viscous diffusivity. When $Re$ is large, most of the fluid doesn’t care much about the frictional effects of its container, and flow has a propensity for disorganization. Using $U = 10^{-4}$ m/s, $L = 2 \times 10^6$ m, and $\nu = 10^{-6}$ m$^2$/s (de Wijs et al. (1998)), we estimate that $Re \approx 2 \times 10^8$ in Earth’s core. Typically, a flow is considered to be strongly turbulent if $Re \gtrsim 10^4$. The Reynolds number can also be seen as a ratio of timescales: the viscous timescale, $L^2/\nu$, versus the advective timescale, $L/U$. The advective timescale is the typical time it would take a parcel of fluid to move across the container, and inertial effects are stronger when this timescale is smaller. The viscous timescale is the estimated time it would take molecular viscosity to calm the flow to a laminar state. A relatively short viscous timescale tells us that viscosity will be an important influence on fluid motions. With $Re \approx 2 \times 10^8$, it would take far too long for friction to pacify flow in the core, and so we give viscosity little regard in this system.

Other important influences on core fluid dynamics are rotation and magnetic field, via the Coriolis and Lorenz forces, respectively. The Rossby number, $Ro = U/\Omega L$, characterizes rotation’s influence by comparing the advective timescale to the angular rotation rate, $\Omega$. One rotation per day on Earth means $\Omega \approx 7 \times 10^{-5}$ rad/s, and so $Ro \approx 10^{-6}$. Thus, it is believed that rotation is of fundamental importance in core convection dynamics. The length scale for which $Ro = 1$ is known as the Rossby radius, and tells us that inertia will become important relative to rotation only on length scales smaller than about ten meters in the core (this is the distance
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Explanation</th>
<th>Definition</th>
<th>Earth’s Core</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rayleigh Number</td>
<td>$Ra$</td>
<td>Buoyancy Diffusion</td>
<td>$\frac{\alpha T g \Delta T L^3}{\nu \kappa}$</td>
<td>$10^{20} - 10^{30}$</td>
</tr>
<tr>
<td>Ekman Number</td>
<td>$E$</td>
<td>Coriolis Viscosity</td>
<td>$\frac{\nu}{2\Omega L^2}$</td>
<td>$\sim 10^{-15}$</td>
</tr>
<tr>
<td>Prandtl Number</td>
<td>$Pr$</td>
<td>Viscous Diffusion</td>
<td>$\frac{\nu}{\kappa}$</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>Prandtl Number</td>
<td>$Pr$</td>
<td>Thermal Diffusion</td>
<td>$\frac{\nu}{\kappa}$</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>Magnetic Prandtl Number</td>
<td>$Pm$</td>
<td>Viscous Diffusion</td>
<td>$\frac{\nu}{\eta}$</td>
<td>$\sim 10^{-6}$</td>
</tr>
<tr>
<td>Magnetic Prandtl Number</td>
<td>$Pm$</td>
<td>Magnetic Diffusion</td>
<td>$\frac{\nu}{\eta}$</td>
<td>$\sim 10^{-6}$</td>
</tr>
<tr>
<td>Reynolds Number</td>
<td>$Re$</td>
<td>Inertia Viscosity</td>
<td>$\frac{UL}{\nu}$</td>
<td>$\sim 10^8$</td>
</tr>
<tr>
<td>Rossby Number</td>
<td>$Ro$</td>
<td>Coriolis Inertia</td>
<td>$\frac{U}{RE} = 2ReE$</td>
<td>$\sim 10^{-6}$</td>
</tr>
<tr>
<td>Chandrasekhar Number</td>
<td>$Q$</td>
<td>Lorenz Viscosity</td>
<td>$\frac{\sigma B_0^2 L^2}{\rho \nu}$</td>
<td>$\sim 10^{14}$</td>
</tr>
<tr>
<td>Elsasser Number</td>
<td>$\Lambda$</td>
<td>Lorenz Coriolis</td>
<td>$\frac{\sigma B_0^2}{2\rho \nu}$</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>Magnetic Reynolds Number</td>
<td>$Rm$</td>
<td>Magnetic Induction</td>
<td>$\frac{UL}{\eta}$</td>
<td>$\sim 100$</td>
</tr>
</tbody>
</table>

Table 1: Nondimensional parameters used to characterize planetary fluid behavior. Dimensional quantities are: $\alpha T$, the coefficient of thermal expansion; $g$, gravitational acceleration; $\Delta T$, the mean temperature contrast; $L$, the length scale of the system; $\nu$, viscous diffusivity; $\kappa$, thermal diffusivity; $\Omega$, angular rotation rate; $\eta$, magnetic diffusivity; $U$, a typical flow velocity; $\sigma$, electrical conductivity; $B_0$, a typical magnetic field strength; $\rho$, mean density; $\eta$, magnetic diffusivity. The estimate given for $Pr$ is that of the thermal Prandtl number of a liquid metal. For compositional convection, which may be important in the core, estimates of a compositional analog of the Prandtl number may be as high as $O(100)$ (Gubbins (2001)).

A fluid parcel will typically travel in a day). As hinted at by the near-alignment of geomagnetic and geographic north poles, rotation is then an essential aspect of
large-scale core fluid dynamics.

Fluids behave very differently when rapidly rotating than not. Rotation acts to organize and constrain fluid motions. The Proudman-Taylor theorem (see section 1.2.2) tells us that under the extreme influence of rotation, fluid motions tend to align in the direction of the rotation axis. This two-dimensionalization of flow fields stands in stark contrast with the isotropic and chaotic nature attributed to turbulent flow. This apparent paradox is the crux of geophysical fluid dynamics. Inertia trumps viscosity, which typically gives rise to strongly three-dimensional fluid motions. But, rotation trumps inertia, and so these fluid motions are reigned in by the Coriolis force. Examining the competition between a fluid motions’ affinity for chaos and rotation’s insistence on order constitutes a large extent of my thesis work.

As mentioned above, there is another important force in play in the core: the Lorentz force. The Lorentz force is the result of the magnetic field acting back on the fluid motions from which it was generated. In the dynamo energy budget, work done by the Lorentz force represents the reduction of kinetic energy as it is being transformed into magnetic energy. A strong magnetic field is thought to influence the motions of electrically conducting fluids in much the same way as does rotation. A parcel of conducting fluid crossing a magnetic field line will bend the field line, creating magnetic tension, which prefers to snap the field line back and the parcel with it. Magnetic field lines resistance to bending tends to two-dimensionalize local flow. This can be seen in the magnetic analog to the Proudman-Taylor theorem (see
section 1.2.2), which tells us that strong magnetic fields will align fluid motions along field lines.

In his seminal linear stability analysis in the 1950s, astrophysicist Subrahmanyan Chandrasekhar (1953, 1954) showed that, when acting separately, both rotation and applied magnetic fields have a stabilizing effect on convection. When acting together, however, this stabilizing effect is offset to some extent, and convective instability occurs more easily when the characteristic Lorenz and Coriolis forces are roughly equal. The ratio between these two forces, respectively, is called the Elsasser number, Λ, which is defined in table 1. Fluid bodies with Λ ≈ 1 are termed magnetostrophic. This linear result is often hypothetically extrapolated to planetary convection; that convection onsets most easily when Λ ≈ 1 leads many to believe that fully-developed convection operates most efficiently in the magnetostrophic regime, and therefore that magnetic field generation will naturally saturate such that Λ ≈ 1 (e.g., Stevenson (2003)). Observations of the geomagnetic field qualitatively support this idea. Downward continuation of magnetic field measurements provide estimates of a 5 Gauss radial field at the CMB. This leads to an Elsasser number Λ ≈ 0.1 in the core. Dynamo theory suggests that the toroidal component of the geomagnetic field may be stronger than its radial component, and so Λ may in fact be closer to unity in the core (Davidson (2001)). A comparison of magnetic field strengths and rotation rates on other planets within our solar system, however, is at odds with this supposition (Stevenson (2003), Olson and Christensen (2006)). For instance, it may be that
the saturation of magnetic field strength is determined by available convective energy, and not rotation rate (Christensen et al. (2009)).

There are then three important ingredients at work in the core: strong convection; strong rotation; and magnetic field. It is my goal to investigate experimentally the interplay between these forces to better understand the fundamental fluid dynamics of dynamo regions.

1.2 Simulating Planetary Core Convection

1.2.1 Rotating Magnetoconvection

There are several different ways to probe experimentally the magnetic field generating regions within planets. Two basic routes are explored: numerical simulations and laboratory experiments. Numerical simulations have the advantage of being capable of dynamically generating magnetic fields via convective flow. In order to achieve this exemplary goal, however, models must invoke unrealistic fluid properties, which may not be relevant to planetary settings. Moreover, numerical simulations are typically incapable of resolving fully developed turbulent (high $Re$) flow, and are often limited to studying weakly supercritical behavior (e.g., Glatzmaier (2002)). Laboratory experiments have the advantage of modeling fluid behavior in liquid metals with inherent fidelity, and are typically capable of reaching much higher Reynolds numbers. Unlike numerical experiments, however, it can be difficult to diagnose fluid behavior in real liquid metals. Furthermore, fully dynamic, self-sustained magnetic
field generation is currently difficult to achieve in the laboratory (e.g., Nataf (2003)).

The difficulty in developing a laboratory dynamo can be seen via the magnetic Reynolds number, which characterizes the ratio between magnetic induction and diffusion (see table 1). In order for fluid motions to generate currents capable of self-sustaining magnetic field generation, $Rm$ must be greater than about $\mathcal{O}(10)$ (Davidson (2001)). Magnetic diffusivity in liquid metal is typically order unity (in units of m$^2$/s), and so $UL > 10$ m$^2$/s is required for dynamo action. Thus, even in an exceptionally large container of liquid metal such as a sphere with a 1 m diameter, fluid velocities must exceed 10 m/s. If we try to drive such a vigorous flow with convection, what type of Rayleigh number would we need? Free fall velocity provides a generous estimate of speed generated by potential energy available to a parcel of fluid by scaling the balance between the inertial and buoyancy terms (see section 1.2.3):

$$U_{\text{free-fall}} = \sqrt{\alpha_T g \Delta T L}.$$  

For $\alpha_T = 10^{-4}$ K$^{-1}$, typical of liquid metals, $g = 10$ m/s$^2$, and $L = 1$ m, $\Delta T$ must be greater than $10^5$ K. This would require at least a gigawatt of power. Those who seek to generate dynamos in the laboratory, therefore, drive the flow with more efficient mechanical techniques such as forced impellers. In doing so, they aim to study magnetic field generation itself, and not necessarily the convection dynamics which are likely important in planetary settings such as Earth’s core.

In contrast, I intend to examine convection dynamics in a laboratory model of planetary conditions. I seek to impose on a container of real fluid (water or liquid metal) various combinations of the key ingredients that make the problem so dynam-
ically rich: convection, rotation, and magnetic field. With this task in mind, much of my Ph.D. work has gone into developing an experimental device capable of producing these three elements, and all in varying degrees of import as characterized by the Rayleigh number, $Ra$, the Ekman number, $E$, and the Chandrasekhar number, $Q$, respectively (see table 1). This is an important point in that in order to extrapolate our results to the extreme conditions within planets, we must understand how the fluid behaves as each of these parameters approaches planetary values.

To accomplish this, we must first understand to what values we want to extrapolate. As mentioned in the introduction, we are capable of indirectly observing that Earth’s core is likely to have a high Reynolds number and a low Rossby number, and we hypothesize/estimate that Earth’s core has an Elsasser number of roughly unity. In order to achieve high $Re$ convective flow, thermal forcing must be strong, and so $Ra$ must be large. In order for $Ro (= ReE)$ to be small, $E$ must be small. And for $\Lambda (= QE)$ to be order unity, $Q$ must then be large. This is the basic premise of the Rotating Magnetoconvection Device (RoMag): I seek to develop an experimental apparatus capable of producing turbulent (high $Ra$), rapidly rotating (low $E$), magnetostrophic (high $Q$) convection.

1.2.2 Governing Equations

As highlighted above, theoretical, experimental, and numerical convection studies must all make simplifying assumptions in order to analyze the complex dynamics
observed in and on planets and stars. Here, I outline the simplifications we employ in
greater detail, and their implications. The equations governing planetary convection
reflect the complex nature of the system. They are the Navier-Stokes or momentum
equation (in a rotating reference frame), the continuity equation, the heat or energy
equation, and the induction equation:

\[
\frac{\partial \rho \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla (\rho \mathbf{u}) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2\rho \mathbf{\Omega} \times \mathbf{u} = -\nabla p + \rho \mathbf{g} + \mathbf{J} \times \mathbf{B} + \rho \nu \nabla^2 \mathbf{u}
\] (1)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\] (2)

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = H_Q + \kappa \nabla^2 T
\] (3)

\[
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B},
\] (4)

where \( \mathbf{u} \) is the fluid velocity, \( \mathbf{\Omega} \) is the planetary (background) rotation vector, \( \mathbf{r} \)
is the displacement vector, \( p \) is pressure, \( \rho \) is the fluid’s density, \( \mathbf{g} \) is gravitational
acceleration, \( \mathbf{B} \) is the magnetic field, \( \mathbf{J} = 1/\mu_0 \nabla \times \mathbf{B} \) is the electric current density,
\( \nu \) is the fluid’s kinematic viscosity, \( T \) is temperature, \( H_Q \) is a heat source/sink, \( \kappa \)
is the fluid’s thermal diffusivity, and \( \eta \) is the fluid’s magnetic diffusivity. Terms in
equation (1) are, from left to right, momentum evolution, advection term, centrifugal
acceleration term, Coriolis term, pressure gradient, gravity term, Lorentz term and
the viscous diffusion term. Terms in equation (3) are temperature evolution, thermal
advection, heat source, and thermal diffusion. Terms in equation (4) are magnetic
field evolution, field advection, induction term, and magnetic diffusion. Here, we
already assume that the fluid is Newtonian, and that molecular diffusivites \( \nu, \kappa, \) and
\( \eta \) are spatially uniform.
The first simplification we will adopt is the Boussinesq approximation, which assumes that fluid density variations $\rho'$ about the mean $\bar{\rho}$ are only important in their contribution to the gravity term $g$ in equation (1). This assumption neglects compressibility effects such as adiabatic cooling that may be important in planetary bodies such as Earth’s core (Anufriev et al. (2005)). Furthermore, we assume that the density variations that contribute to the buoyancy term are due only to first-order thermal expansion, $\rho' = -\alpha T' \bar{\rho}$, where $\alpha$ is the fluid’s coefficient of thermal expansivity, and $T'$ is the anomalous temperature, $T' = \bar{T} - T$, where $\bar{T}$ is the mean temperature. The gravity term is now $g = g + g'$, and $g'$ is the anomalous buoyancy $g' = g (\rho'/\bar{\rho})$. Since the mean (hydrostatic) gravitational force and centrifugal acceleration are both conservative, they can be absorbed into the pressure gradient term as $P = p + \bar{\rho} g x_g + \frac{1}{2} \Omega^2 s^2$, where $x_g$ is the displacement vector parallel to $g$ and $s$ is the distance from the rotation axis. The Navier-Stokes and continuity equations are now:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho} \nabla P - \alpha T' g + \frac{1}{\rho} J \times \mathbf{B} + \nu \nabla^2 \mathbf{u}$$

(5)

$$\nabla \cdot \mathbf{u} = 0;$$

(6)

and overbars have been dropped from the mean density $\rho$.

Next, we will simplify the geometry and thermal configuration by reducing the problem to one of Rayleigh-Bénard convection. The fluid volume is reduced to a horizontal layer between a relatively cold upper boundary and a relatively hot lower boundary. The gravity vector is now vertically downward, uniform and constant,
\[ g = -g \hat{z}, \] such that the imposed thermal boundary conditions are destabilizing. The thermal boundary conditions are \[ T' = \Delta T/2 \text{ at } z = 0 \] and \[ T' = -\Delta T/2 \text{ at } z = h, \] \text{h} being the height of the layer, and any further sources of heat (\( H_Q \) in equation (3)) are no longer considered. Furthermore, the rotation vector is assumed to be vertical.

The governing equations are then:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \hat{z} \times \mathbf{u} = -\frac{1}{\rho} \nabla P + \alpha_{T'} g \hat{z} + \frac{1}{\rho} \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{u} \quad (7)
\]

\[
\nabla \cdot \mathbf{u} = 0 \quad (8)
\]

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T \quad (9)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}, \quad (10)
\]

where \( \hat{z} \) is the vertical unit vector.

To nondimensionalize the equations we adopt a length scale \( L \sim h \), an advective timescale \( \tau \sim L/U \), where \( U \) is a typical velocity (\( \mathbf{u} \sim U \hat{u} \)), and we scale temperature and magnetic field strength by \( \Delta T \) and \( B_0 \), respectively. We scale pressure as \( P \sim \rho \Omega U L \hat{P} \) and current density as \( \mathbf{J} \sim B_0 \mu_0^{-1} L^{-1} \), where \( \mu_0 \) is the permeability of free space. Thus, equations (7,9,10) can be re-written in terms of nondimensional (hatted) terms:

\[
\frac{U^2}{L} \frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \frac{U^2}{L} \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} + 2\Omega U \hat{\mathbf{z}} \times \hat{\mathbf{u}} = -\Omega U \hat{\nabla} \hat{P} + \alpha_{T'} g \Delta T \hat{\mathbf{z}} + \frac{B_0^2}{\rho \mu_0 L} \hat{\mathbf{J}} \times \hat{\mathbf{B}} + \frac{\nu U}{L^2} \nabla^2 \hat{\mathbf{u}} \quad (11)
\]

\[
\frac{\Delta U}{L} \frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\Delta U}{L} \hat{\mathbf{u}} \cdot \nabla \hat{T} = \frac{\kappa \Delta T}{L^2} \nabla^2 \hat{T} \quad (12)
\]

\[
\frac{B_0 U}{L} \frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \frac{B_0 U}{L} \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{B}} = \frac{B_0 U}{L} \hat{\mathbf{B}} \cdot \nabla \hat{\mathbf{u}} + \frac{\eta B_0}{L^2} \nabla^2 \hat{\mathbf{B}} \quad (13)
\]

We complete non-dimensionalization by dividing equation (11) by \( \Omega U \), equation
(12) by $\kappa \Delta T/L^2$, and equation (13) by $\eta B_0/L^2$. We are then left with the following non-dimensional governing equations:

\[
Ro \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + 2\hat{z} \times u = -\nabla P + \frac{2RaE}{RePr} T\hat{z} + \Lambda J \times B + 2E \nabla^2 u
\]

\[
\nabla \cdot u = 0
\]

\[
P_c \left( \frac{\partial T}{\partial t} + u \cdot \nabla T \right) = \nabla^2 T
\]

\[
Rm \left( \frac{\partial B}{\partial t} + u \cdot \nabla B \right) = Rm B \cdot \nabla u + \nabla^2 B,
\]

where the ‘hats’ have been dropped. The Rossby number, $Ro$, Rayleigh number, $Ra$, Ekman number, $E$, Reynolds number, $Re$, Prandtl number, $Pr$, Elsasser number, $\Lambda$, and magnetic Reynolds number, $Rm$, are non-dimensional parameters defined in table 1, and $Pe = RePr$ is the Peclet number. Note that to switch to a spherical shell geometry, the only necessary change is in the buoyancy term, wherein vertical gravity $g\hat{z}$ becomes radial $g\hat{r}$.

1.2.3 Scaling Analysis

As mentioned in the introduction, it is thought that when $Ro \ll 1$, a fluid will be strongly influenced by rotation. Since $U$ is not known a priori in experiments and simulations, neither is the Rossby number, $Ro = U/\Omega L$. The convective Rossby number, $Ro_c$, circumvents this difficulty by employing the free-fall velocity assumption. Free-fall velocity arises by assuming that, in convectively driven flows, inertia scales
with buoyancy, yielding:

\[ U_{\text{free-fall}} = \sqrt{\alpha_T g \Delta T L}, \]  

(18)

By assuming \( U = U_{\text{free-fall}} \), the Rossby number becomes the convective Rossby number,

\[ R_o_c = \sqrt{\alpha T g \Delta T / 4 \Omega^2 L} = \sqrt{Ra E^2 Pr^{-1}} \approx \text{buoyancy}/\text{Coriolis}. \]  

(19)

The convective Rossby number is often used to determine an \textit{a priori} estimate of the importance of rotation in convection experiments and models, where \( Ra, E, \) and \( Pr \) are control parameters, but \( U \) is not known in advance (e.g., Julien et al. (1996b), Liu and Ecke (1997), Stevens et al. (2009), Zhong et al. (2009)).

Free-fall velocity characterizes the perfect conversion of available potential energy to kinetic energy. As such, \( U \leq U_{\text{free-fall}} \) and thus \( R_o \leq R_o_c \). Often, researchers find that the convective Rossby number overestimates the measured Rossby number by about an order of magnitude or more (e.g., Christensen (2002)). The buoyancy term in equation (14) scales as

\[ \frac{2 Ra E}{Re Pr} = \frac{2 R_o_c^2}{R_o}, \]  

(20)

such that if \( R_o_c \approx R_o \), buoyancy will scale with \( R_o \), as expected.

When \( R_o, E, \) and \( \Lambda \) are small, the order one terms in equation (14) are the Coriolis and pressure gradient terms. The dynamical balance between these,

\[ 2 \mathbf{\hat{z}} \times \mathbf{u} \sim -\nabla P, \]  

(21)

is known as geostrophy. If we take the curl of equation (21) and use equation (15),
we arrive at the Proudman-Taylor theorem:

\[ \frac{\partial u}{\partial z} = 0. \]  \hspace{1cm} (22)

Thus, in this geostrophic limit, we expect flow to be invariant along the axis of rotation. In a realistic container, however, it is impossible for convection to transport heat without breaking the Proudman-Taylor theorem. Slight departures from geostrophic flows are called quasigeostrophic, and here the Proudman-Taylor theorem becomes the Proudman-Taylor constraint, which describes a rapidly rotating system’s proclivity toward axially invariant flow structures.

Flow structures which are roughly axially invariant will necessarily extend across the container in the axial direction. What length scale do we expect in the perpendicular direction? To answer this question, we must include another term in our balance: the viscous term. The question then becomes, at what length scale can viscosity balance the Coriolis term in the horizontal direction? If we take the curl of the geostrophic balance with the viscous term included, we arrive at

\[ 2\Omega \frac{\partial u}{\partial z} \sim \nu \nabla^2 \omega, \]  \hspace{1cm} (23)

where \( \omega = \nabla \times u \) is the vorticity. The z-component of equation (23) is

\[ \frac{\partial u_z}{\partial z} \sim \frac{\nu}{2\Omega} \nabla^2 \omega_z \sim \frac{\nu}{2\Omega} \frac{u_h}{l_h^3}, \]  \hspace{1cm} (24)

where \( u_h \) is a typical horizontal velocity and \( l_h \) is a typical horizontal length scale. Assuming \( \partial u_z / \partial z \) and \( u_h \) are order one (e.g., Stellmach and Hansen (2004)), we can
determine the typical horizontal length scale,

\[ l_h \sim E^{-1/3} L. \tag{25} \]

This scaling estimate has been verified for quasigeostrophic convection, where flow is often manifested as tall, thin columnar structures (e.g., Busse and Carrigan (1976)).

As mentioned above, rotation and magnetic fields are thought to have similar effects on flow, in that they both act to two-dimensionalize flow structures. This can be seen in the magnetic analog to the Proudman-Taylor constraint. Let us impose on a conducting fluid a uniform, vertical magnetic field, \( B_z \). In the limit of strong, invariant \( B_z \), the dominant term in the induction equation is the induction term:

\[ \mathbf{B} \cdot \nabla \mathbf{u} = B_z \frac{\partial \mathbf{u}}{\partial z} = 0. \tag{26} \]

This again means that strong magnetic fields will tend to two-dimensionalize flow, as

\[ \frac{\partial \mathbf{u}}{\partial z} = 0. \tag{27} \]

Though by different means, we have thus produced the same result as through the Proudman-Taylor theorem. The imposition of both rotation and vertical magnetic fields will constrain fluid motions to tall, thin structures, and thereby inhibit convection. In his linear stability analysis, Chandrasekhar (1954) shows that despite this constraining influence of each force acting alone, when acting together, the restriction on length scales is somewhat released, and so the suppression of convection’s onset is somewhat relaxed. As mentioned in the introduction, this depression of the critical
Rayleigh number for instability is often extrapolated to planetary settings, wherein magnetic field growth is thought to saturate naturally such that these two forces are in global equilibrium (e.g., Stevenson (2003)). This hypothesis, however, has not been tested in turbulent convection. RoMag was designed and built with this task in mind.

Much of the work presented here focuses on the efficiency with which fluid motions transfer heat, as characterized by the Nusselt number. The efficiency of convective heat transport is an important aspect of the energy budget of convective dynamo systems (Labrosse (2003), Nimmo (2009)). Furthermore, the Nusselt number provides a global observable that is intimately linked to turbulent fluid motions. The Nusselt number is defined as

\[ \text{Nu} = \frac{\text{Total Heat Transfer}}{\text{Conductive Heat Transfer}} = \frac{qL}{k\Delta T}, \]  

where \( q \) is the heat flux, \( L \) is the container’s length scale, \( k \) is the fluid’s thermal conductivity, and \( \Delta T \) is the total temperature drop. Thus, \( \text{Nu} = 1 \) for purely conductive heat transfer, and higher \( \text{Nu} \) corresponds to more efficient convective heat transfer. Typically, more vigorous flow means higher \( \text{Nu} \).

Heat transfer behavior will be modulated by the strong influences of rotation and magnetic fields. As fluid motions are organized by Coriolis or Lorentz forces, their degrees of freedom are limited, and the efficiency of convective heat transport will then be decreased relative to that of non-rotating, non-magnetic convection. In this work, I focus on the influence of rotation on convective heat transport in comparison to that without rotation as a gauge of rotation’s import. In order to understand this
comparison, we need to look first at heat transfer by non-rotating convection.

![Figure 2: A schematic of well mixed Rayleigh-Bénard convection. Temperature is shown as white solid lines. The interior fluid is isothermal, and half of the total temperature drop $\Delta T/2$ occurs in the thermal boundary layers, of thickness $\delta_k$. If the thermal boundary layers are thin $L \gg \delta_k$ and do not communicate, then $Nu \propto Ra^{1/3}$.](image)

Heat transfer in non-rotating Rayleigh-Bénard convection has been studied for decades. Often, these studies seek to understand how heat transfer efficiency scales with convective driving, $Nu \propto Ra^\alpha$. The classical analytical prediction follows from boundary layer arguments, and yields $\alpha = 1/3$ (e.g., Kraichnan (1962)). Here, it is assumed that the bulk of the convecting volume is well mixed (isothermal), and is
sandwiched between two quasi-static fluid layers at the top and bottom boundaries, called the thermal boundary layers. A schematic of this configuration is shown in figure 2. The overall conductive heat flux is $q_{\text{conductive}} = k\Delta T/L$. In the idealized quasi-static boundary layer, heat transfer is entirely conductive, and an isothermal interior means half the total temperature drop occurs in each (top and bottom) boundary layer. Since the total vertical heat flux must be the same through all horizontal planes, the heat flux through the boundary layer equals the total heat flux, $q_{\text{Total}} = k\Delta T/2\delta_\kappa$, where $\delta_\kappa$ is the thermal boundary layer thickness. The Nusselt number is then

$$Nu \equiv q_{\text{Total}}/q_{\text{conductive}} \approx L/2\delta_\kappa.$$  \hspace{1cm} (29)

We can further assume that the boundary layers do not communicate with each other. The boundary layer thicknesses $\delta_\kappa$ are then independent of the total layer thickness $L$. Then $Nu \propto Ra^\alpha \Rightarrow L \propto (L^3)^{1/3}$, and $\alpha$ must be 1/3. This is the classical theoretical heat transfer scaling for turbulent Rayleigh-Bénard convection.

There have been several other theoretical scalings developed, including $\alpha = 1/4$ and $\alpha = 1/2$ (e.g., Kraichnan (1962), Spiegel (1971)). The $\alpha = 1/2$ scaling is hypothesized to occur in what is called the ultimate regime, where convection is supposed to be so vigorous that boundary layers cannot survive. This scaling has not been observed unambiguously (Glazier et al. (1999)). The $\alpha = 1/4$ scaling is thought to be important in turbulent convection in liquid metals (Rossby (1969)). It is thought that in high $Re$ convection in low $Pr$ fluids (such as liquid metals), the dominant
convective force balance is between the inertial, buoyancy and viscous terms. This triple-balance is accomplished by scaling buoyancy with viscosity, where the velocity is scaled as the free-fall velocity (equation (18)). Thus,

\[ \alpha_T g \Delta T \approx \nu \nabla^2 U \sim \nu \sqrt{\alpha_T g \Delta T L} \delta^2, \] (30)

where \( \delta \) is the length scale below which viscosity is important. By assuming this length scale is the thermal boundary layer length scale, \( \delta \approx \delta_\kappa \approx L/Nu \), we can solve for a \( Nu-Ra \) scaling:

\[ Nu \sim \left( \frac{Ra}{Pr} \right)^{1/4} \] (31)

Early convection studies seemed to support either the classical \( \alpha = 1/3 \) (e.g., Dropkin and Globe (1959)) scaling or the \( \alpha = 1/4 \) (e.g., Rossby (1969)). More recent experiments, however, find scalings between 1/3 and 1/4, notably nearer to \( \alpha = 2/7 \) (Castaing et al. (1989), Takeshita et al. (1996), Glazier et al. (1999)). This is discussed in further detail in section 3.1.

In the presence of strong rotation and/or magnetic fields, our theoretical description of heat transfer becomes murkier still. Arguably, there are no generally accepted theoretical heat transfer scalings in these systems.
2 The Rotating Magnetoconvection Device:

Experimental Anatomy
In this chapter, I describe the Rotating Magnetoconvection Device (RoMag). First, I briefly describe the experimental apparatus in section 2.1. I then provide a more detailed description of the device and experimental techniques.

Section 2.2 discusses the general context of the experimental design. Section 2.3 describes the experimental framework and rotation system. Section 2.4 details the
heating and cooling systems of the device. Section 2.5 explains the storage and filling processes of working fluids. Section 2.6 discusses the sensory and data acquisition system.

Section 2.7 illustrates a typical sequence of operations for a convection study using RoMag. Section 2.8 presents the experimental capabilities of the device in terms of the non-dimensional parameters accessible to RoMag. Finally, section 2.9 introduces the collaboration between experimental work and numerical simulations conducted in this study.

2.1 RoMag: an Overview

The rotating magnetoconvection device (RoMag) was designed to span broad ranges of the control parameters $Ra$, $E$, and $Q$, which make up our three core ingredients. It is for this reason that a plane layer device was chosen, as convection experiments in spherical shells are limited to a narrow range of rotation rates ($E$) such that they correctly modify effective gravity to match the shell geometry (Sumita and Olson (2003)). Furthermore, the plane layer geometry simplifies an already complex rotating magnetoconvection system, and allows us to compare directly with the wealth of thermal turbulence studies conducted in plane layer geometries. Our convection setup then consists of a cylindrical convection tank that sits atop a rotating pedestal and can be surrounded in full by a solenoidal electromagnet (figure 3). The convection tank is 20 cm in diameter, and has variable height, from 1.3 cm to 20 cm. Increasing
Table 2: Thermophysical properties of water (20°C) (Lide (2000)), sucrose solution (14.4\% sugar by mass, 20°C) (Hirst and Cox (1976)), and gallium (30°C) (Okada and Ozoe (1992)).

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Units</th>
<th>Water</th>
<th>Sucrose Solution</th>
<th>Gallium</th>
</tr>
</thead>
<tbody>
<tr>
<td>density</td>
<td>( \rho )</td>
<td>kg/m(^3)</td>
<td>1000</td>
<td>1050</td>
<td>6095</td>
</tr>
<tr>
<td>thermal expansivity</td>
<td>( \alpha_T )</td>
<td>K(^{-1})</td>
<td>(2 \times 10^{-4})</td>
<td>(8 \times 10^{-4})</td>
<td>(1.3 \times 10^{-4})</td>
</tr>
<tr>
<td>viscous diffusivity</td>
<td>( \nu )</td>
<td>m(^2)/s</td>
<td>(10^{-6})</td>
<td>(1.6 \times 10^{-6})</td>
<td>(3.2 \times 10^{-7})</td>
</tr>
<tr>
<td>thermal diffusivity</td>
<td>( \kappa )</td>
<td>m(^2)/s</td>
<td>(1.4 \times 10^{-7})</td>
<td>(1.4 \times 10^{-7})</td>
<td>(1.3 \times 10^{-5})</td>
</tr>
<tr>
<td>magnetic diffusivity</td>
<td>( \eta )</td>
<td>m(^2)/s</td>
<td>—</td>
<td>—</td>
<td>0.2</td>
</tr>
<tr>
<td>Prandtl number</td>
<td>( Pr = \nu/\kappa )</td>
<td>—</td>
<td>7</td>
<td>10</td>
<td>0.026</td>
</tr>
<tr>
<td>magnetic Prandtl number</td>
<td>( Pm = \nu/\eta )</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>(1.6 \times 10^{-6})</td>
</tr>
<tr>
<td>specific heat</td>
<td>( Cp )</td>
<td>J/kg K</td>
<td>4180</td>
<td>3815</td>
<td>398</td>
</tr>
<tr>
<td>thermal conductivity</td>
<td>( k )</td>
<td>J/m s K</td>
<td>0.6</td>
<td>0.6</td>
<td>31</td>
</tr>
<tr>
<td>electrical conductivity</td>
<td>( \sigma )</td>
<td>siemens/m</td>
<td>—</td>
<td>—</td>
<td>(3.88 \times 10^6)</td>
</tr>
</tbody>
</table>

the tank’s height increases the length scale, \( L \), and so increases \( Ra \), decreases \( E \), and increases \( Q \), all else being constant (note, though, that \( \Lambda \) is length scale independent). Working fluids are water, sucrose solution, and gallium, whose thermophysical properties are listed in Table 2, and the temperature dependence thereof is given in section 2.5.1. The cylinder sidewalls are polycarbonate (for use with water), or stainless steel (for use with gallium). Top and bottom endwalls (called thermal blocks) are either aluminum, teflon coated aluminum, or tungsten coated copper, the latter two being developed specifically for use with the corrosive liquid gallium.

Underneath the bottom thermal block is a non-inductively wound electrical resistance element that provides the heat power to drive convection. Electrical power is delivered to the heater from a 5 kW power supply through a system of heavy gauge
electrical slip rings. Heat is removed from the system by a liquid water heat exchanger (cooling block) located above the top thermal block. Thermostated water is pumped from a 10 kW precision chiller into the rotating frame via a two-channel rotary union. The aluminum cooling block is comprised of two double-wound spiral flow channels for thermal uniformity.

The convection tank setup, and lower instrument table, are rotated by a brushless (for rotation rate uniformity) servomotor capable of 10 Nm torque. The motor is able to rotate our experiment at more than 100 rpm, or 1.7 Hz. The rotation rate is uniform to better than 0.5%.

The magnetic field is applied via a 450 kg solenoidal electromagnet. The magnet has a hollow 40 cm diameter cylindrical inner bore, and is held above the convection tank by a lift system so the magnet can be lowered to surround the convection tank. The solenoidal coils are wound such that the magnet generates a uniform, vertical field within the convection tank to within 0.5%. The magnet is capable of imposing fields up to 1300 Gauss (0.13 T). The magnet is powered by a 16 kW power supply, and both magnet and power supply are liquid cooled.

2.2 Spirit of RoMag

As mentioned in section 2.1, the basic idea behind RoMag’s design is the combination of three major ingredients of core flow: turbulent convection (heat source); rotation; and magnetic field. Although some of these components have been instituted exper-
imentally in the past, they have not all been realized simultaneously. The difficulty faced in developing the experimental device lies mostly in the combination of these components.

2.2.1 The Magnet

In order to impose a magnetic field strong enough for Lorentz forces to be within the desired range, currents of nearly 100 Amps are needed. Currents of this strength must be passed through heavy gauge copper wire, and even then will generate significant heat through electrical resistance. The magnet, then, needs to be actively cooled via recirculation of thermostated water. Thus, in order to reach the magnetostrophic regime, a rather large, self-contained electromagnet is required. Our magnet has a mass of 450 kg.

The need for a large magnet is not an overwhelming difficulty per se, but in combination with rapid rotation, design complications amass. Rotating a 450 kg magnet at up to 100 RPM is a dangerous prospect. We overcome this challenge by arranging the coils of the magnet in such a way that they generate a nearly uniform, vertical magnetic field within the volume of the convection tank. This allows us to leave the magnet in the stationary lab frame while the experiment rotates through it’s field. If the field is azimuthally uniform, there will be no difference between a stationary and co-rotating magnet.

An illustration of the magnet is shown in figure 4. It was fabricated by Walker
Figure 4: A schematic of RoMag’s electromagnet. Units of length are given in inches, as per magnet design.

LDJ and has basic dimensions 0.61 m wide by 0.61 m deep by 1.07 m high with a 0.38 m diameter hollow inner bore. The magnet has a lower base plate designed to be fitted to jack screws and a roller guide so that it can be lowered from above to encapsulate the convection tank within its inner bore (see section 2.3). The magnet coils are arranged in an upward spiraling hour-glass shape, with coil density increasing near the top and bottom of the magnet. This current distribution generates a magnetic field that is vertical within the magnet’s bore, and uniform to within 0.5% within the experimental volume. Figure 5 shows measurements of the uniformity of field strength
Figure 5: Radial profiles of experimental magnetic field strength within the solenoid's hollow bore.

within our working volume. The magnet is powered by a Walker Scientific HS 200-80 power supply with a PSC-3 control module. With maximum current output of 80 A, the magnet produces a 0.13 T field (see figure 6). Both the magnet and power supply need to be liquid cooled by at least 3 gallons per minute (GPM) chilled water.

2.2.2 Other Design Considerations

Another complication introduced by the desire for a strong, uniform magnetic field is that we now require the experimental device to be constructed from non-magnetic
Figure 6: Experimental magnetic field strength versus amperage.

materials. In the presence of an applied field, magnetic materials will distort this field, and in so doing alter the influence of that field on our convecting fluid. Non-magnetic here then refers to a negligible amount of paramagnetism, which can be described by a material’s permeability relative to a vacuum, $\mu/\mu_0$. Non-magnetic materials have $\mu/\mu_0 \approx 1$. Steel is a classic example of a strongly magnetic material that is commonly used in experimental fabrication, with $\mu/\mu_0 = 700$. This construction limitation
restricts our use to non-magnetic materials such as aluminum ($\mu/\mu_0 = 1.000022$) and copper ($\mu/\mu_0 = 0.999994$) alloys, and plastics such as Teflon (PTFE). In most high-heat applications, aluminum and copper alloys are best suited. Unfortunately, we must also account for the corrosive nature of liquid gallium in our experimental design. Both copper and aluminum react with gallium. Stainless steel (SS 316L) provides a good (albeit expensive), non-magnetic, non-reactive, high temperature alternative. All of these materials are used in different capacities in the construction of RoMag, and will be discussed further below.

One last basic design consideration I will mention before discussing the individual systems in detail is the on-board diagnostics. Information is passed from the rotating frame to the static lab frame via electrical slip rings (see section 2.6). Previous experiments have found difficulty in maintaining a high signal to noise ratio in certain sensors (such as acoustic Doppler transducers) when passing these signals through slip-rings, on account of the introduction of rotational noise. For this reason, RoMag has been designed to acquire data within the rotating frame. This requirement leads to the incorporation of a co-rotating lower turntable to which many of our diagnostics systems are attached in order to preserve the quality of our measurements.

2.3 Musculoskeletal System

The musculoskeletal system of RoMag refers to its basic structure and mechanical framework. RoMag’s framework consists of a lower frame and an upper frame. Both
are constructed from structural aluminum and painted with protective enamel to guard against corrosion by gallium.

The lower frame is shown in figure 7, and has dimensions of 0.86 m wide by 0.78 m deep by 0.61 m high. The lower frame supports the weight of all experimental components, and is leveled manually using thin metal shims. At the center of the top
of the lower frame is a platform on which sits our bearing, which separates the rotating and stationary experimental components. The bearing is an MTO-145 Super Heavy Duty Turntable manufactured by the Kaydon Corporation, Inc., has dimensions 0.05 m \( h \times 0.3 \text{ m OD} \times 0.145 \text{ m OD} \), and is rated to bear a 68,000 kg load capacity. The top surface of the bearing supports a co-rotating stainless steel pedestal on which sits the convection tank. A hollow shaft is fixed to the large inner bore of the bearing, which permits the passage of fluid, electrical power and electrical signals from below to the convection tank along the device’s rotation axis (see sections 2.4 and 2.6). The bottom of this shaft supports the co-rotating lower table, which allows certain diagnostic and control systems to reside in the rotating frame. The lower table is constructed from aluminum and has a diameter of 0.81 m. The top of the lower frame is amalgamated into one water-tight surface using plexiglass to prevent any convection tank spills from reaching the electronics on the lower table.

Rotation is driven by a brushless Kollmorgen Goldline XT servomotor, model MT502B2-R1C1. The motor can rotate with torques up to 10.1 Nm, and is connected to the bearing via a driving belt (with adjustable tension) attached to a 40:1 ratio gear head. The motor is controlled and monitored by a Kollmorgen Servostar CD servodrive, model CR06250. We connect to servodrive via serial cable using a Dell laptop and Kollmorgan software, which allows us to control the rotation rate of the motor, and therefore 1/40 that of the bearing, lower table, and pedestal assembly. The servodrive also monitors the rotation rate and outputs a voltage signal that we
incorporate into our diagnostic system (see section 2.6). Typical steady rotation rates are uniform to better than 0.5%, as seen in figure 8.

The upper frame is also constructed from aluminum, and is intended to provide support to the magnet, which sits atop the experimental device. With the addition of the upper frame, RoMag stands at 3.3 m tall. A photograph of the assembled frame (with pedestal and magnet lift assembly) is shown in figure 9.

When not running MHD experiments, the magnet is held above the convection
tank. When magnetic fields are desired, the magnet is lowered to surround the convection tank. This basic setup is shown in figure 10. The weight of the magnet is supported by the lower frame, and the upper frame provides lateral support. The magnet is raised and lowered by a lift system shown in figure 11. Three adaptor feet are affixed to the base of the magnet and threaded into 1.5 m stainless steel
Figure 10: The two positions of RoMag’s electromagnet. The schematic shows the magnet in the up position (right), and in its active down position (left). The 450 kg solenoid is supported by three jack-screws that sit on the lower frame, is raised and lowered by a 3/4 hp gearmotor that turns the jack-screws, and is supported laterally by RoMag’s upper frame.

The magnet is then raised and lowered by turning the jack-screws in one direction or the other. This is accomplished by a 3/4 hp Duff-Norton gear motor (model 817MDSF), rated at 33 Nm maximum torque. The three jack-screws are linked by three worm gear actuators and two mitre boxes. Safety components of the lift system include a dynamic brake and a rotary limit switch. Raising and lowering
of the magnet are aided by a long Delrin guide shaft, which stabilizes the magnet via guide rollers.

### 2.4 Circulatory System

The circulatory system of RoMag refers to its heating and cooling system, comprised of both electrical power and coolant circulation.

Heat is required to drive convection. In order for the system to equilibrate ther-
Figure 12: A schematic of RoMag’s circulatory system. Heat is generated by electrical currents. That heat is removed by a system of thermostated, recirculating water reservoirs. Liquid cooling is indicated by dashed lines.

In summary, all heat injected must be removed. This is the underlying principle of RoMag’s circulatory system: we aim to provide efficient conduits of energy flow beginning at our electrical power source and ending in the atmosphere. Figure 12 shows a schematic flow chart of this path. Heat is generated by passing currents through electrical resistors. This essentially occurs in three RoMag components: the convection tank’s heat pad; the magnet power supply; and the magnet itself. In the former, heat is intentionally generated in an electrical resistor; in the latter two, heat generation is a byproduct of the large currents required to generate strong magnetic fields.

The heater is a mica resistance element, manufactured by Minco (model HM6833).
The heater’s conductors are non-inductively wound, i.e. arranged such that they induce negligible magnetic field. This is important in maintaining a uniform magnetic field within the convection tank. The heater has an electrical resistance of 6833 Ohms. A direct current is passed through the heater by way of an Argantix XDS 300-17 power supply. This power supply is capable of delivering up to 300 V at 16.7 A, or up to 5010 W of power. With the power supply, we control the voltage output. It is useful then to recognize that the input heat power, $P$, will be

$$P = \frac{V^2}{R} = \frac{V^2}{6833 \text{ Ohms}}, \quad (32)$$

where $V$ and $R$ are voltage and resistance, respectively. The heater’s power supply, which is powered by three-phase 208 VAC facility power, is in the stationary lab frame. The heat pad, however, is in the rotating experimental frame.

This heating current, then, must be passed through electrical slip-rings into the rotating frame. The slip-rings are manufactured by Poly-Scientific (of Northrup Grumman), model AC6098-24. The slip rings have a 0.2 m outer diameter, and a 0.1 m diameter hollow inner bore. This allows the slip-rings to be situated outside the rotating shaft, between the lower rotating table and the main bearing. The solid state slip-rings can maintain connection at up to 250 RPM. They consist of six power rings that can each carry up to 50 A, and 54 additional signal rings that carry no more than 10 A, and are meant for low-voltage sensor signals (see section 2.6).

The heat produced by our heating element flows up into and through the convection tank (see section 2.5), above which it must be removed. This is accomplished by
Figure 13: An illustration of RoMag’s cooling block.

a liquid cooled aluminum heat exchanger, which is referred to as the cooling block, and is shown in figure 13. The cooling block is a cylindrical block of aluminum (T6061) into which has been cut two double wound spiral flow channels. Water is forced into the block at the inlet pipe connection and flows through either the top (figure 13) or bottom (not shown) channel. Here, it spirals inward and clockwise until it reaches the center of the cooling block after which point it spirals back outward and counterclockwise. This double spiral flow path is devised such that as the water flows through the cooling block and heats up, the general pattern of temperature will be roughly uniform. This, together with a thin (3/16”) copper bottom cover, allows
us to treat the bottom surface of the cooling block as nearly isothermal.

Water is forced into the cooling block by a CP-75 pump in our Thermo NESLAB HX300 precision chiller, hereon called the lab chiller. The lab chiller maintains a reservoir of fluid (we use water) at a thermostated temperature to within 0.1°C. As in the case of the heater power supply, here we have to deliver chilled water from a source in the lab frame to the cooling block, which resides in the rotating frame. In order to accomplish this, we utilize a fluid rotary union. We use a two-channel rotary union manufactured by Rotary Systems (model 12-N-23212).

Figure 14 shows a schematic of the heat delivery and removal system to the experimental apparatus. Direct currents are sent through electrical slip-rings to the mica heat pad. Heat is then removed by forcing chilled water through the rotary union and into the heat exchanging cooling block. This heat is then deposited in the lab chiller. From here, heat must be transferred to its final hub, the rooftop chiller.

The rooftop chiller is an air cooled General Air recirculating water chiller. It has been installed on the roof, directly above our lab. Supply and return water lines have been hard-lined through the ceiling. From here, the water circuit is split into three lines, for each of the rooftop chiller’s three thermal loads. These loads are: the lab chiller, the magnet power supply, and the magnet itself. Each of these thermal loads requires at least 3 GPM of recirculating cool water, which is easily met by the rooftop chiller’s internal pump, and monitored by mechanical flowmeters. The returning, warmer water is cooled by a compressor, releasing RoMag’s residual heat.
Figure 14: A schematic of RoMag’s experimental thermal circulation system.

to the atmosphere.

When all three components are at maximum power, we should be generating less than 20 kW of excess heat. This is roughly the equivalent of an idling automobile.
Figure 15: A schematic of RoMag’s convection tank stack. From bottom to top are the pedestal, leveling platform, insulator, heater, bottom thermal block, sidewall, top thermal block, cooling block, and expansion tank.
2.5 Gastrointestinal System

The gastrointestinal system of RoMag refers to the convection tanks, and the storage, transport, and experimental implementation (convection tank filling) of the working fluids. Working fluids are those whose behavior we examine within the convection tank. They include (but are not necessarily limited to) water, sucrose solution, and gallium.

2.5.1 Fluid Properties

The thermophysical properties of the working fluids are dependent on their temperature. The fluid properties of the three fluids, water, sucrose solution, and gallium, are described here as a function of temperature. The fluid properties needed for the measurements made in this study are density ($\rho$), thermal expansivity ($\alpha_T$), viscosity ($\nu$), thermal diffusivity ($\kappa$), thermal conductivity ($k$), and specific heat ($C_P$). The units used for each are:

\[
\begin{align*}
\rho &\rightarrow \text{kg/m}^3 \quad (33) \\
\alpha_T &\rightarrow \text{T}^{-1} \quad (34) \\
\nu &\rightarrow \text{m}^2/\text{s} \quad (35) \\
\kappa &\rightarrow \text{m}^2/\text{s} \quad (36) \\
k &\rightarrow \text{W/m}^2\text{K} \quad (37) \\
C_P &\rightarrow \text{J/kgK} \quad (38)
\end{align*}
\]
The properties of water, in the units given above, are (Lide (2000)):

\[ \rho = 999.8 + 0.1041T - 9.718 \times 10^{-3}T^2 + 5.184 \times 10^{-5}T^3 \]  

(39)

\[ \alpha_T = -6.82 \times 10^{-5} + 1.70 \times 10^{-5}T - 1.82 \times 10^{-7}T^2 + 1.05 \times 10^{-9}T^3 \]  

(40)

\[ \kappa = 1.31210^{-7} + 6.97210^{-10}T - 5.63110^{-12}T^2 + 2.63310^{-14}T^3 \]  

(41)

\[ k = 0.5529 + 2.66210^{-3}T - 2.37410^{-5}T^2 + 1.10810^{-7}T^3 \]  

(42)

\[ C_P = \frac{k}{\rho \kappa}, \]  

(43)

where \( T \) is the temperature of the fluid in °C.

The properties of sucrose solution have an added dependency of the concentration of sugar. By mass, the ratio of sugar to water in the solution is referred to as degrees Brix, °BX. The density of the solution is measured prior to conducting experiments, and variations about that density with temperature are assumed to be determined by the thermal expansivity of pure water given above. The remaining fluid properties of sucrose solution are (Hirst and Cox (1976)):

\[ \nu = \frac{6.581}{(((61.5 + T) - (1 + 0.011 T)^°BX)^2)/\rho} \]  

(44)

\[ \kappa = \frac{k}{\rho C_P} \]  

(45)

\[ k = 0.5758 + 1.360 \times 10^{-3}T - 3.006 \times 10^{-3}°BX - 2.511 \times 10^{-6}°BX T - 3.341 \times 10^{-6}°BX^2 - 1.182 \times 10^{-7}°BX^2 T \]  

(46)

\[ C_P = [1 - (0.632 - 0.001 T)^°BX/100]4184. \]  

(47)

Although the results presented in this document do not include convection experiments in liquid metal, the thermophysical properties of gallium are included for
completeness (Okada and Ozoe (1992)):

\[ \rho = 6330 - 0.7717T \]  
(48)

\[ \alpha_T = 0.7717/\rho \]  
(49)

\[ \nu = (1.207 \times 10^{-2} - 5.754 \times 10^{-5} T + 7.891 \times 10^{-8} T^2)/\rho \]  
(50)

\[ \kappa = \frac{k}{\rho C_P} \]  
(51)

\[ k = -7.448 + 0.1256 T \]  
(52)

\[ C_P = 397.6, \]  
(53)

and here \( T \) is the fluid temperature in degrees Kelvin.

2.5.2 The Tank

The convection tank is the belly of the experiment. The convection tank sits atop the pedestal in what we call ‘the stack’. The stack is essentially everything above the pedestal that co-rotates. Figure 15 shows a schematic of stack components. Just above the top of the pedestal is a separate stainless steel platform with three leveling screws between it and the pedestal, such that the tank can be leveled independently of RoMag’s frame. Above this platform sits our bottom insulator, a simple right cylinder constructed from CS85, an insulating material with thermal conductivity \( k = 0.31 \) W/Km and has an upper operating temperature limit of 1000°C. Above this insulator sits the heat pad, which is bolted to the bottom of the bottom thermal block using a thin stainless steel backing plate. The bolt pattern for the heat pad attachment system is drilled into the top of the insulating block so the heat pad is always fixed.
and centered relative to the rest of the stack. Above the bottom thermal block is the sidewall, and above that is the top thermal block. Together, these three components make up the convection tank, which contains the working fluid. Vacuum grease is applied to the outer diameter of the sidewall near it’s top and bottom edges in order to assure that the o-ring embedded in the thermal blocks achieves an adequate seal. Above the convection tank sits the cooling block (discussed in section 2.4). On top of the the cooling block is the expansion tank, discussed below, and several layers of closed foam insulation to minimize thermal communication with the environment. The stack is mechanically secured by eight stainless steel rods that are threaded into the bottom platform and tightened from above the cooling block with stainless steel nuts and washers.

The convection tank is comprised of top and bottom caps, called thermal blocks, into which fits a cylindrical sidewall. The sidewall is essentially a piece of 20 cm diameter tubing, cut to a variety of different heights. We have three bottom thermal blocks, four top thermal blocks, and about twenty different sidewalls.

The sidewalls are cut to different heights so that we can change the fundamental length scale of the system, and therefore have access to different (scale-dependent) parameter ranges. For example, the Rayleigh and Ekman numbers are dependent on the length scale to the third and second power, respectively. Changing the tank height is therefore our most effective method of spanning large ranges of these parameters. Several different materials are also used in sidewall construction. Typically, we use
clear polycarbonate tubing for water experiments that allow us to see into the tank, which is important for tank filling procedures and experimental flow visualization. In gallium, an opaque fluid, we use instead stainless steel sidewalls, with which we can reach higher temperatures, as the plastic sidewalls will begin to deform near 60 C and will melt at temperatures above 100 C.

Another reason for using sidewalls of different material for the different working fluids is due to the thermal conductivity. We would like to treat the sidewall as a perfect insulator. This is important for two reasons: first, heat escaping outward from the sidewalls gives us a height-dependent Nusselt number, which generates a number of dynamical and technical concerns; second, heat traveling upward through the sidewall should be negligible relative to that transferred by convection. The former concern is ameliorated by adding additional layers of closed-foam or fibrous insulation batting outside the tank (non-toxic Insulfrax insulation is used for high temperature applications as a safe alternative to fiberglass). The latter issue is problematic when undertaking low Nu experiments in water. The ratio of heat passed vertically through the sidewall \( Q_{\text{sidewall}} \) to that through the fluid \( Q_{\text{fluid}} \) is:

\[
\frac{Q_{\text{sidewall}}}{Q_{\text{fluid}}} = \frac{k_{\text{sidewall}} A_{\text{sidewall}}}{\overline{Nu} k_{\text{fluid}} A_{\text{fluid}}},
\]

(54)

where \( k \) is thermal conductivity of each medium, \( A \) is the area of a horizontal cross-section of each component, and \( Nu \) is the Nusselt number. Note that \( k_{\text{fluid}} Nu \) can be treated as an effective thermal conductivity of a convecting medium. Using stainless steel, \( k_{\text{sidewall}} = 16 \text{ W/mK} \), and water, \( k_{H_2O} = 0.6 \text{ W/mK} \), as well as \( A_{\text{sidewall}} = \)
\[ \pi(4''^2 - (4 - 1/8)^2) = 3.1''^2 \] for the 1/8'' thick sidewalls, and \( A_{\text{fluid}} = \pi((4-1/8)^2) = 47.2''^2 \) for the fluid layer, the heat ratio is

\[ Q_{\text{sidewall}}/Q_{H_2O} = 1.75 Nu^{-1}, \] (55)

between stainless steel sidewalls and water. This is a \( Nu \) dependent effect, and will be more significant at lower \( Nu \). When seeking \( Nu \propto Ra^a \) scaling laws, the lower \( Ra \) data will have an anomalously low \( Nu \) (relative to the case of insulating sidewalls). We would then measure an \( \alpha \) that is too low. Therefore, stainless steel sidewalls should not be used for convection in water. In gallium, however, \( k_{\text{fluid}} = 31 \) W/mK, and so the stainless sidewalls will have a much weaker effect:

\[ Q_{\text{sidewall}}/Q_{Ga} = 0.034 Nu^{-1}. \] (56)

The thermal blocks, top and bottom, make up the other two components of the convection tank. The thermal blocks have smooth, flat surfaces that interface with the working fluid. Six holes are drilled radially inward from the sides of each thermal block for temperature probes. These holes, seen in figure 16 are equally spaced in azimuth, and are located just 3 mm from the fluid surface. The bottom thermal block has a filling port through which we fill and empty the convection tank. The top thermal blocks have two additional ports, one for tank overflow and decompression during filling and draining procedures, and another to be connected to an expansion tank. The expansion tank is a stainless steel tank that is attached to the top of the cooling block and permanently open to overflow from the convection tank. The
Figure 16: A design drawing of the bottom thermal block. The location of the bottom thermistors is shown. Placement of the top thermal block thermistors is identically situated relative to the fluid layer. The thermistors are equally spaced in azimuth, and offset by 3 mm from the fluid.

expansion tank gets its name from its purpose: it serves as a reservoir for excess working fluid resulting from thermal expansion during convection experiments. This allows the convection tank to be effectively open to atmosphere so that pressurization issues don’t arise, and assures that the convection tank, once full, will remain so despite thermal expansion and contraction of the working fluid as mean temperatures fluctuate. The top thermal block also houses internal temperature and velocity sensors (discussed in section 2.6).
As in the selection of sidewall material, the design of the top and bottom thermal blocks also holds conductivity ratios as a main concern. A basic assumption of Rayleigh-Bénard convection is that the top and bottom fluid boundaries are isothermal. In real experiments, the bounding surfaces have finite conductivity, $k_{\text{surface}}$, which allows temperature variations to form. The Biot number, $Bi$, characterizes the ‘isothermality’ of the boundary in a heat transfer problem by comparing the temperature drop in the boundary to that of the interior (or, more accurately, compares the total effective conductance of the layers, which is equivalent to temperature drop assuming conservation of vertical heat flux on horizontal planes). Low $Bi$ systems will have smaller thermal variations in the boundaries than in the interior. The Biot number is defined as

$$Bi = h \frac{D_{\text{surface}}}{k_{\text{surface}}},$$

(57)

where $h$ is the overall heat transfer coefficient, $D_{\text{surface}}$ is the boundary thickness, and $k_{\text{surface}}$ is the boundary’s thermal conductivity. Thus, for convection,

$$Bi = \frac{k_{\text{fluid}}}{k_{\text{surface}}} \frac{D_{\text{surface}}}{D_{\text{fluid}}} N_u.$$  

(58)

We assume that our heater and cooling block represent isothermal surfaces, and the bounding surfaces in question are then our top and bottom thermal blocks. We also assume that $Bi \leq 0.1$ corresponds to negligible temperature gradients in the boundary. Above this value, convective behavior can be influenced significant thermal heterogeneity in the boundaries. It can be seen in equation 58 that $Bi$ will be largest for tall tanks of gallium. The goal of our convection tank design is to maximize our
available Nu under the constraint that $Bi \leq 0.1$ for convection in liquid gallium. There are two ways to do this according to equation 58: maximize the boundaries’ thermal conductivity, and minimize the boundaries’ thickness.

Thus we have designed and constructed thermal blocks from copper ($k_{copper} = 390$ W/mK), and we have made them as thin as possible. For the bottom thermal block, the limiting factor in how thin the block can be made is in the filling port, which becomes too small for reasonable fill times below 1/8” NPT (national pipe threading). We designed the block to be only 0.6” (0.015 m) thick, which allows Nusselt numbers as high as 16.4 with $Bi \leq 0.1$ for a 20 cm tank of gallium.

The top thermal block is a bit more tricky. The need for a top thermal block to house temperature and velocity sensors brought us to design two separate top blocks, one with portholes for sensors and another, thinner block without. The thinner block has thickness 0.52” (0.013 m), made possible by using smaller 1/16” NPT filling ports. This together with the 3/16” thick cover on the cooling block allows us to reach $Nu \leq 13.9$ for $Bi \leq 0.1$. The second top thermal block must house the diagnostic sensors, and so must be both considerably thicker, and must have non-negligible void space for the probes, further reducing its effective conductance. With this block, whose design is shown in figure 20 and has total thickness 1.85” (0.047 m), we can reach $Nu \leq 5.7$ for $Bi \leq 0.1$.

Another difficulty faced in thermal block design for use with gallium is due to the corrosive nature of the liquid metal. Gallium reacts with copper. As such, we
seek a coating material to prevent the gallium from being in direct contact with the copper thermal blocks. An obvious choice would be a coating application of the notoriously inert material, Teflon. Unfortunately, this material is such an efficient thermal insulator that even a micron-thick coating would have a negative impact on the Biot number calculations above. Thus, we have chosen to coat the copper with a plasma-spray application of tungsten, often used to prevent corrosion and wear on aerospace parts. This was done by the Plasma Coating Corporation of Gardena, CA. Tungsten has been shown not to react significantly with gallium, and has thermal conductivity of 170 W/mK, such that a 0.008” (0.2 mm) thick coating will have a negligible impact on the calculations above.

Convection experiments in water require less restrictive thermal block design. Here, we use any combination of the copper blocks described above, and four more made from aluminum (T6061, $k_{\text{aluminum}} = 167$ W/mK).

2.5.3 Storage and Filling

Water storage is somewhat simple. We use tap water for our experiments, as water in its purer (deionized and distilled) forms is more corrosive. We keep a holding tank full of water at a high point in the lab. This tank is loosely sealed, to allow the water to degas while preventing significant evaporation. Degassing water is important because otherwise the fluid will release gas when heated in the convection tank, and this will generate bubble formation at the top of the convection tank. In Rayleigh-
Bénard convection, the tank’s top and bottom surfaces must be completely wetted. The formation of bubbles, or a less than completely filled tank forms an unintentional and effective obstacle for heat transfer. Much of the convection tank filling process is developed to ensure proper wetting of the top boundary.

The convection tank is filled with water by way of gravitational pressure and a system of rubber tubing and plastic quick-disconnect tube fittings and valves. Prior to filling, the tank is tilted so that the overflow port is near the high-point. This forces the air in the tank to be forced out of the system to avoid bubble formation. Alternatively, bubble formation can be avoided by filling a tank that has been evacuated. Tank evacuation is accomplished using a modified bike tire pump. By reversing the plunger and check valve of a bike pump, I apply negative pressure to the top of the expansion tank. This is especially effective when the convection tank is partially full, so that any air trapped in the filling tubes will be evacuated as well. In order to maintain the vacuum, a well sealed experiment is required. A pressure valve is affixed to the top of the expansion tank to avoid building positive pressure in the closed system. If the water has not been completely degassed, introducing it to less-than-atmospheric pressure will force further degassing. This is counter-productive in the quest for perfect wetting, and so tank evacuation is most often used in the gallium filling procedure.

Gallium is stored in a 2 gal (7.57 L) stainless steel Binks pressure vessel. Hard-wired to the vessel is a thermostated electrical heating system so that gallium, which
freezes just above room temperature (\(\sim 30 \, ^\circ\text{C}\)), can be melted and maintained in liquid form when in use. The vessel is also connected to a tank of Argon, which is inert and slightly heavier than air, such that oxidization of gallium is limited. Oxidized gallium is ‘cleaned’ in a fume hood by way of hydrochloric acid solution, which reacts with the oxides to release oxygen.

The convection tank filling procedure for gallium is complicated by its susceptibility to freezing at room temperature. While filling the tank with gallium, crystallization of the liquid metal means a blockage in the line, which allows a buildup of pressure that has a potentially catastrophic (and embarrassing) outcome. (When running experiments with liquid metal, I make sure to cover all floor drains in the event of a catastrophic spill.) Thus, before filling the convection tank with gallium, the tank and all fill tubing is preheated to \(\sim 40 \, ^\circ\text{C}\). Gallium is capable of super-cooling, provided is is prohibited from being in contact with a suitable thermal energy store. For this reason, Teflon fill tubing and other insulating plumbing components are used for gallium filling. The tubing system is preheated by cycling warm water. The convection tank is preheated using a combination of the heater and cooling block (which can be set to cycle 40 °C water), and monitored using the thermal diagnostic system (see section 2.6). Again, the tank is tilted so as to force out any air when filling, and the tank can again be evacuated. Positive pressure must be generated at the storage vessel. This is accomplished either by opening the regulator valve between the argon tank and pressure vessel, or by way of a displacement pump and a short
length of rubber tubing. (Note that a displacement pump is an especially dangerous place for crystals to form, as the crystal and pump will conspire to tear the tube, bleeding out the liquid metal). Filling progress can be observed by monitoring the tank’s thermistor signals. Temperature and temperature variance will change when a thermistor comes into contact with the fluid. When full, overflowing gallium can be collected using a separate tube and container. Some excess gallium should be injected into the tank with the expansion tank valve open, such that thermal contraction will not leave an under-filled convection tank. Bubbles can be detected by tilting the tank and observing thermal signals. When the fluid is convecting, the thermal signal near a bubble will show considerably lower variance, as the bubble acts as a lowpass thermal filter. A self-regulating tape heater is affixed to the expansion tank and associated plumbing. Freezing gallium there would allow thermal expansion to produce a pressure excess, promoting catastrophic leakage.
Figure 17: A schematic of RoMag’s convection nervous system. Experimental control and monitoring occurs in the lab frame. Data, however, is acquired and stored in the rotating frame during experimental operation. Not shown are visualization systems and Hall probe magnetometers.
2.6 Nervous System

The nervous system of RoMag refers to its electronic controls, sensors and data acquisition systems used to regulate experiments and diagnose fluid behavior. There are two main sensor groups: thermal measurements, and acoustic doppler velocimetry. We have also developed visualization methods and magnetometers for experimental use, of which there will be limited discussion here. A schematic of the basic structure of RoMag’s nervous system is show in figure 17.

Most measurements are made through the SCXI-PXI system, which essentially allows us to make time series measurements of voltage drops across up to 64 circuits. The SCXI box is a National Instruments SCXI-1000 with two 1303 amplification modules and one 1300 excitation module. These modules convert analog voltage signals into digital readings. The digital readings are then transmitted by SCXI cable to the PXI box, which is a National Instruments PXI 1042 with a 8186 embedded controller and a 6061E input/output module. The PXI box is basically a data acquisition PC, and on it we run National Instruments Labview software to collect, monitor, and record our data at rates of higher than 1000 Hz. The SCXI-PXI system permits multiplexing measurements occurring at equally spaced time intervals and simultaneously on all channels. Equally spaced time intervals are optimal for spectral analysis using Fourier transform methods. Simultaneous measurements are important for comparative statistics such as cross-correlations.
2.6.1 Experimental Control

The electronic controls system is rather basic. Most experimental components (heater, chillers, magnet) are run using front-end control modules, and their behavior is monitored using our more advanced diagnostic systems.

The heater output power is measured in what we call the shunt box, which contains a current shunt and a voltage divider. The current shunt allows us to measure the current flowing to the heat pad, $I$, by channeling it through a resistance element of known resistivity, and measuring the resulting voltage drop. The voltage divider channels the current through a resistor of known resistivity in parallel, which allows us to measure the voltage drop across the heater, $V$. Heat power supplied to the convection tank is then

$$P_{\text{heater}} = IV. \quad (59)$$

The servomotor is controlled using Kollmorgen Servostar software on a PC that is connected by serial cable to the servodrive. The servodrive outputs a voltage proportional to the measured rotation rate of the servomotor, which we send through the slip-rings into the rotating frame so we can record the rotation rate using the SCXI-PXI system.

2.6.2 Thermometry

Thermal measurements are the work-horse of RoMag’s nervous system. The thermometry system can be broken down into its three basic measurements: top and
bottom boundary temperatures, internal fluid temperatures, and heat flux. All three of these measurements require accurate temperature sensors.

Temperature measurements are made using thirty-eight custom made GE Thermometrics sheath thermistors and two YSI 410 pipe thermistors. Thermistors use resistance elements (here, ceramic) whose resistivity is temperature dependent. A third-order approximation of the relationship between temperature and resistivity is given by the Steinhart-Hart equation:

\[
\frac{1}{T} = a + b \ln(R) + c \ln^3(R),
\]

where \( R \) is the resistivity of the thermistor. A thermistor’s resistivity can be measured by passing a small known current through it and measuring the resulting voltage drop. This equation can be used to solve for temperature as a function of resistivity, provided the coefficients \( a, b, \) and \( c \) are known. These coefficients are particular to each thermistor, and can be solved for by measuring the resistivity of each thermistor at three known temperatures. This is accomplished by immersing all of the thermistors in the thermostated reservoir of a precision chiller. I measure the resistivities of each thermistor for \( \sim 30 \) minutes at 10 Hz. This is done for each of three temperatures, \( T_1, T_2, \) and \( T_3, \) spread across an expected range of RoMag operating temperatures. The resistivity measurements are averaged in time to get three values \( R_1(n), R_2(n), \) and \( R_3(n) \) for each of \( n \) thermistors, corresponding to the three different temperatures. The coefficients for each thermistor \( (a, b, \) and \( c) \) can then be solved for in MATLAB using the Steinhart-Hart equation.
The SCXI 1300 excitation module sends a known (small) current through each thermistor, and measures the resulting voltage drop, which can be converted into a resistivity. This resistivity measurement is converted into measurements of temperature by the Labview software on the PXI box by using the coefficients determined for each thermistor. Thus, the data acquisition software outputs measurements of temperature for each thermistor.

Thermistor calibrations are tested by again immersing the thermistors in an isothermal bath, this time at a temperature different from $T_1$, $T_2$, and $T_3$. Another, more comprehensive test is done before running experiments. When the convection tank is full and insulated, the heater is turned off and the chiller is set to a temperature near ambient. After sufficient time, the system should thermally equilibrate and the thermistors should all read the same temperature. These tests show that my calibrations are good to better than $\pm 0.01$ K. The quality of the insulation can be tested by fixing the cooling block temperature higher or lower than ambient. If the tank is perfectly insulated, one should again observe constant temperatures throughout the stack after the system has equilibrated.

At present, the most useful measurements made in RoMag are the Nusselt and Rayleigh numbers. These parameters require essentially two dynamic measurements: mean temperature drop; and total heat flux. The mean temperature drop is calculated as the difference between the average bottom and top thermistor measurements. The heat flux measurement is made in two ways: by using the electrical power supplied to
the heater (discussed above), and by measuring the temperature change and flow rate through the cooling block. Increasing the temperature of a volume of water (with density $\rho$ and heat capacity $C_P$) by $\Delta T_{\text{cool}}$ increases its internal energy by $\rho C_P \Delta T$.

If the volume flux of water through the cooling block is $\Phi_{H_2O}$, then the heat power absorbed by the recirculating coolant is simply

$$P_{\text{cool}} = \rho C_P \Delta T_{\text{cool}} \Phi_{H_2O}. \quad (61)$$

This calculation requires that we measure $\Delta T_{\text{cool}}$ and $\Phi_{H_2O}$. The temperature drop is measured by way of the pipe thermistors, which are embedded into the input and output fluid lines of the cooling block. The flow rate is measured using a Proteus 4000 turbine flowmeter (model 04006BN9-D), which outputs a voltage signal that increases with the speed of the turbines, and therefore increases with the flow rate. The flowmeter was calibrated in house using a series of totalizer measurements outlined below.

We use the lab chiller to fill a 15 gallon barrel at a constant flow rate (estimated a priori based on Proteus digital readout) all while recording the Proteus output voltage. We repeat 22 times at 10 different flow rates, and plot flow rate ($=15$ gallons/fill time) versus voltage to calibrate the Proteus voltage output. We start with the chiller off, but with the outflow valve open to a predetermined position for desired flow rate. The chiller supply-side hose empties into a 15 gallon barrel and the return-side hose is connected to an external water source to replenish chiller reservoir. We begin recording Proteus voltage via Labview, and then turn on the
chiller pump. The chiller quickly (nearly instantaneously) reaches the intended flow rate and remains constant to within 1%. When the water level in the barrel reaches the 15 gallon mark, we shut off the chiller pump. We estimate our error in total volume to be 1/16 gallon (based on the thickness of the indicating line on the barrel), or 0.4% of 15 gallons. The resulting voltage time signal has a boxcar shape. An example of this time signal is shown in figure 18.

For each run we calculate mean voltage, \( \langle V \rangle \), and the area under the curve, \( \int V \, dt \), between chiller pump start and stop times. The time it takes to fill our 15 gallon barrel is \( \frac{\int V \, dt}{\langle V \rangle} \). The flow rate corresponding to voltage \( \langle V \rangle \) is then \( \Phi = 15 \) gal. \( \langle V \rangle / \int V \, dt \). Thus, we calculate and plot \( \Phi \) versus \( \langle V \rangle \) for each of our 22 trials. A quadratic curve
fit yields calibration curve:

$$\Phi = -0.13838V^2 + 2.636V - 0.57624,$$

in units of gallons per minute (GPM). The resulting plot and its quadratic regression are shown in figure 19.

To test our calibration curve, we executed several more 15 gallon totalizer trials but now with time-varying flow rates. Scaling the Proteus output voltage by equation 62, and again integrating under the curve, we calculate the total flow volume for each test, which all fall within 0.3% of 15 gallons. This shows that the flow rate measurement is well calibrated.

The accurate measurement of $\Delta T_{\text{cool}}$ and $\Phi_{H_2O}$ permit accurate measurements of
Figure 20: A design drawing of the top thermal block. Eight probe holes are situated for allowing thermistor and doppler access to the internal fluid. Dimensions are shown in inches.

heat flux passing through the top of the stack. We compare this heat flux measurement to that made at the heater \( P_{heat} \). As we maintain a nearly ambient mean temperature for most of our experiments (especially in water experiments), the heat lost to (or gained from) the room is negligible, and therefore the difference between \( P_{heat} \) and \( P_{cool} \) is negligible. As such, we typically use \( P_{heat} \) as our heat flux for calculating the Nusselt number.

Up to eight thermistors can be inserted into the convection tank through the top thermal block. The planform arrangement of possible thermistor locations is
shown in the top thermal block drawing in figure 20. These thermistors have varying length, and can then be placed within the tank at varying vertical locations $h/2 \leq z < h$, where $h$ is the tank height. Typically, the central thermistor is placed at the very center of the tank, for comparison with other thermal turbulence studies. These thermistors have stainless steel sheaths of only 0.9 mm thick. This allows measurements high frequency temperature statistics within the fluid.

2.6.3 Doppler Velocimetry

Visualization techniques for fluid dynamics experiments abound. However, liquid metals are opaque, a quality that renders almost all visual diagnostics ineffective. Flow measurements can be made in gallium using the relatively new acoustic Doppler velocimetry technique. Although results from Doppler measurements are not shown in the results presented in this document, a description of the development of this experimental technique is included here for completeness.

At the heart of acoustic velocimetry is the piezoelectric transducer, which is basically a small-scale speaker/microphone. Probes containing piezoelectrics send high frequency sonar pulses into the fluid, and ‘listen’ for backscattered acoustic signals. Assuming we know the speed of sound in the fluid, the time delay between the initial chirp and the incident backscatter gives the position of the fluid parcel in question. The probe’s sounding is repeated at high frequency. Moving fluid parcels will scatter energy that is well correlated with that from the last iteration. Maximizing auto-
correlations in space shows how far the parcel has moved since the last chirp. This permits measurement of velocity as a function of time, along a one-dimensional beam. The velocity measured is also one-dimensional, co-linear with the orientation of the acoustic beam.

We use a Met-flow Ultrasonic Velocity Profiler (UVP) Duo MX Doppler system, with custom made probes operating at 2, 4, and 8 MHz. The Doppler signals are acquired in the rotating frame using the UVP Duo MX (called the Doppler box in figure 17). The Doppler system is controlled by a PC in the lab frame through an ethernet line. I found that the most reliable method of ethernet communication between lab and rotating frames to be through the power slip-rings, using a commercially available powerline ethernet adaptor (NETGEAR XEB1004).

Acoustic energy is reflected by imperfections in the medium that present a local impedance to the sound wave. In gallium, these imperfections are naturally occurring oxides. In water, the scattering particles must be seeded in the fluid. We use polystyrene microspheres, shown in figure 21. The particles are typically between 50 and 100 µm in diameter, and have density \( \sim 1.050 \text{ g/cm}^3 \). It is important that the seed particles are neutrally buoyant in the fluid medium. For this reason, we add sugar to water to match the solution’s density to that of the seed particles. Empirically, we find we need 144.4 g of sugar per L of water for \( \rho_{\text{solution}} = 1.050 \text{ g/cm}^3 \) (corresponding to 14.4 Brix).

Stokes’ Law describes how fast spherical bodies fall through viscous fluids, assum-
Figure 21: Microsphere acoustic seed particles in suspension in sucrose solution in the 5 cm convection tank. Sugar is added to water to match the density of the particles, so that they remain in suspension.

Using laminar flow (Stokes (1851)):

\[ v = \frac{1}{18} \left[ \frac{(\rho_s - \rho_f)gd^2}{\mu} \right], \]  

(63)

where \( v \) is settling velocity, \( \rho_s \) is the density of the solid sphere, \( \rho_f \) is the fluid density, \( g \) is gravitational acceleration, \( d \) is the sphere’s diameter, and \( \mu \) is the dynamic fluid viscosity, \( \mu = \nu \rho_f \), where \( \nu \) is the kinematic viscosity. Thus,

\[ v = \frac{1}{18} \left[ \frac{\delta \rho gd^2}{\nu} \right], \]  

(64)

where \( \delta \rho \) is the density difference fraction \( (\rho_s - \rho_f)/\rho_f \). The time it takes for the
Figure 22: Density mismatch ratio versus seed particle settle time for laminar flow. Density matching capabilities cannot keep particles in suspension for a typical thermal equilibrium time (~3 hours). Turbulence is required to keep the fluid well seeded.

particles to fall from suspension in the 5 cm tank can be estimated as

\[
t_{\text{settle}} = \frac{0.05m}{v} = \frac{9.2}{\delta \rho} \text{ seconds}
\]  

(65)

So settling time is inversely proportional to the error in density matching sugar water to particle density. Thus, if the fractional density difference is ten percent, the particles will all settle out within a minute and a half. If the mismatch is one percent, it will take fifteen minutes. This relationship is shown in figure 22. A typical thermal equilibrium time for convection experiments is about three hours. In order to keep the seed particles in suspension for three hours in a laminar fluid, a density matching
of better than one part in a thousand is required. Fortunately, the presence of turbulence in most of our convection runs will keep the microspheres adequately stirred for the duration of a typical experiment.

The Doppler probes are cylindrical (resembling crayons in shape and size), and are made from either Delrin (a plastic) or stainless steel (for high temperature applications). We situate our probes either pointing downward in one of the probe holes shown in figure 20, or pointing horizontally through the sidewall. The orientation of sidewall probes is shown in figure 23. Two basic probe orientations are possible: a radial profile, intersecting with the tank’s center, and a cord profile, at angle of 42°
from the radial direction. The probe holders are machined from stainless steel, and are affixed to the stainless steel sidewalls using a silver weld. Welded joints between probe holders and sidewalls are coated in acrylic nail polish to protect the silver from gallium corrosion.

Doppler velocimetry is tested by way of impulsive spin-up. Spin-up (or spin-down) refers to the transient adjustment of a fluid from one state of solid body rotation to another. The control parameters here are

\[
\text{Ekman number } = E = \frac{\nu}{2\Omega h^2} = \frac{\text{viscous}}{\text{Coriolis}}; \quad \text{Rossby number } = \epsilon = \frac{\Omega_2 - \Omega_1}{\Omega}, \tag{66}
\]

where \(\nu\) is the fluid’s kinematic viscosity, the length scale, \(h\), is the cylinder height, \(\Omega\) is the background rotation rate, and \(\Omega_1\) and \(\Omega_2\) are the initial and final rotation rates, respectively. Most often, \(\Omega\) is set to be the larger of \(\Omega_1\) and \(\Omega_2\) (save for the early linear spin up work by Greenspan and Howard (1963) who used \(\Omega = \Omega_1\)). The Ekman number is usually assumed to be small \((E \ll 1)\), and the Rossby number will fall in the range \(-1 \leq \epsilon \leq 1\), (e.g. \(\epsilon > 0\) for spin-up and \(\epsilon = 1\) is spin-up from rest).

When the tank is impulsively spun-up, the fluid will, at first, remain in it’s initial solid body rotation state, now rotating past the probe at a rate of \(\epsilon\Omega\). In the case of linear spin-up \((\epsilon \ll 1)\), the solution for the evolution of the azimuthal velocity is (Greenspan (1968)):

\[
u_{\phi} = -\epsilon\Omega r \exp \left(-2E^{\frac{3}{2}}\Omega t\right), \tag{67}
\]

where \(t\) is time after the impulse.

We measure this spin-up process with our cord profile Doppler probe. Figure 24
shows the geometry of the cord profile. Doppler velocimetry, again, measures co-
linear velocity, which corresponds to $u \cdot \hat{x}$ in Figure 24. If we assume a solid body
rotation flow structure $u_{sb} = \Omega \hat{r} \hat{\phi}$, our probe will ‘see’ $u_{sb} \cdot \hat{x} = \Omega r \cos \theta = \Omega r \hat{y} = \Omega y$. So we expect to measure linear spin-up velocities that are constant along this chord profile.

Figure 25 shows instantaneous velocity data along the chord profile shortly after
the impulse. The relatively constant velocity measurement as a function of distance
from the probe verifies that the probes can ‘see’ to the other side of the tank. Figure 26
shows raw velocity data at four points spanning the chord profile along with the
theoretical curve from equation 67. The agreement between the measurement and
theory verifies the accuracy of the doppler velocimetry technique in sucrose solution,
even for slow fluid velocities. Spin-up tests like these can be performed just prior to
and just after taking data in a convecting fluid in order to verify the validity of the
Doppler data.
Figure 25: Raw spin-up velocity data from an instantaneous doppler cord profile measurement in a 5cm tank of sucrose solution, 5 seconds after impulsively spinning up the container. $E = 5 \times 10^{-5}$, $\epsilon = 0.1$. The dashed, vertical line indicates the far side-wall boundary of the tank along this cord.

2.6.4 Flow Visualizations

Visualizations are acquired in water experiments by using a Watec 902H Ultimate video camera in the laboratory frame. The camera is connected to a laptop via an analog-to-digital converter, and is captured using BTV Pro software. Still images may be sampled from the video stream, examples of which are shown in figure 41 on page 110. The fluid is injected with a Kalliroscope solution, which contains small, flake-like particles that stay in suspension and tend to align with shear structures in
Figure 26: Raw spin-up velocity time series in sucrose solution. Each channel corresponds to a distance from the probe (100 - 4cm, 200 - 7.7cm, 300 - 11cm, 400 - 14.5cm). $E = 5 \times 10^{-5}$, $\epsilon = 0.1$, $h = 5cm$. The solid line represents the theoretical prediction from equation (67).

The convection tank is illuminated by way of a laser light sheet. This is a laser beam which has been sent through a cylindrical lens that spreads the light into a two-dimensional planar sheet. We orient the light sheet to shine vertically through the center of the tank. The Kalliroscope solution must be dilute enough that the laser light reaches the far end of the tank, yet rich enough to provide sufficient contrast
for imaging. More concentrated Kalliroscope solutions permit observations of fluid behavior just inside the sidewall, as seen in figure 15.

2.7 Sequence of Operations

Here, I outline a typical sequence of operations for running convection experiments with RoMag, and the precise methodology by which the essential measurements presented in this document are made.

Starting with a fully assembled stack, the thermal data acquisition system is initiated so the fluid can be monitored. The tank is filled with the working fluid (see section 2.5.3) and insulated. Once the tank is full and insulated, the heating system is initiated by activating the roof chiller, lab chiller, and hear power supply. These must be activated and verified in this order, so heat is not trapped anywhere in the circulatory system (e.g., figure 12). A rotation rate is chosen to achieve the desired Ekman number based on the tank size in use. Rotation is slowly ramped up (~25 RPM/second) to this desired rate using the servo control software (section 2.6.1). Solid body rotation should be achieved within several minutes (e.g., figure 26).

Next, a heat power is chosen in order to reach a desired flux Rayleigh number. The flux Rayleigh number is defined as

$$Ra_f = RaNu = \frac{\alpha_T q L^4}{\rho C_p \kappa^2 \nu},$$

(68)

where $q$ is the heat flux, which is the heat power per unit area. With RoMag, I control the heat flux, not the temperature drop across the fluid. I do not, then, fix
the Rayleigh number, but rather the flux Rayleigh number. In order to approach a desired Rayleigh number with this device, the relationship between $Nu$ and $Ra$ must be known a priori. The lower limit on heat power is determined by our ability to measure small heat fluxes through the cooling block (see section 2.6.2). Below about 5 W, the difference between the heat power measured through the cooling block begins to differ significantly from the input heat power. I do not report results for experiments with heat fluxes of less than 10 W. The upper bound for heat power is either 5 kW, which is the maximum output of the power supply, or determined by the limit of our maximum allowable temperature drop, $\Delta T$. Our cooling block can be fixed to be no colder than about 5 °C (in order to avoid freezing issues in the lab chiller). As $\Delta T$ increases, so does the peak temperature of the system. We limit the maximum temperature based on the weakest thermal component. For example, polycarbonate begins to lose structural integrity above about 60 °C. The maximum heat power is then dependent on the maximum $\Delta T$, which is dependent on the materials present and the convecting fluid’s ability to transfer the heat upward.

Once a heat power value is chosen (typically, I start at the lowest value), the current is delivered to the heater. The response of the system is monitored thermally using the Labiew software. Typically, the system takes approximately three hours to equilibrate. I consider the system thermally equilibrated when no trend is observed on any thermistor over a time period of 30 minutes. Once the system has equilibrated, I begin recording data at a typical rate of 10 Hz for about an hour. Once recording is
finish, I change the heat power and wait for the system to re-equilibrate. Typically, I step logarithmically from the lowest to highest heating value so that I acquire approximately 15 $Nu - Ra$ cases for each Ekman number in each tank. Once these are acquired, I switch the rotation rate and again step through several heat power points. Convection cases are occasionally duplicated to verify repeatability. Once all desired Ekman numbers are acquired, I shut down the device, beginning with the servomotor, then the heater, followed by the lab chiller, rooftop chiller, and data acquisition system. I remove the insulation, drain the tank, and disassemble the stack. I then rebuild the stack with a sidewall of a different height, and repeat.

The data taken are as follows. Temperature time series are acquired from six thermistors in the bottom thermal block (figure 27a), six thermistors in the top thermal block (figure 27b), up to eight thermistors within the tank, and two thermistors measuring the input and output temperatures of the cooling block. Beyond this, several voltage time series measurements are acquired: from the current shunt and voltage divider to measure the power supplied to the heater; from the Proteus flowmeter to measure the flux of coolant through the cooling block; and from the servodrive to measure the rotation rate of the experiment. These data are stored in a text file by the Labview software on the PXI box for each convection case, and then transferred to an external hard drive. The data are processed using MATLAB software on a MacBook Pro.

Essentially, six separate quantities are discussed in this document that are cal-
culated from these data for each convection case. They are the Prandtl, Ekman, Rayleigh and Nusselt numbers, and the temperature cross-covariance and central temperature variance measurements.

Four of these parameters are explicitly dependent on the physical properties of the fluid: the Prandtl, Ekman, Rayleigh, and Nusselt numbers. The fluid properties change with fluid temperature (section 2.5.1). In order to avoid large changes in the fluid properties, I try to keep the experiment near room temperature ($\sim 20^\circ C$). When I step up the power applied to the convection tank for each case, I also decrease the set point temperature of the lab chiller, and therefore decrease the temperature of the cooling block, in order to compensate for the growing temperature drop across the tank. This is an inexact tuning, but maintains nearly room temperature (18-22 $^\circ C$) until the highest heat powers are reached. For most cases, then the fluid properties are fairly well-behaved. As the temperature drop across the fluid layer becomes larger than about 30$^\circ C$, the mean temperature of the fluid will increase, changing (slightly) the properties of the fluid.

Using measurements from the top and bottom thermistors, the mean temperature of the fluid is calculated as

$$T_{\text{fluid}} = \frac{1}{12} \left( \sum_{i=1}^{6} T_{i}^{\text{top}}(t) + \sum_{i=1}^{6} T_{i}^{\text{bottom}}(t) \right),$$

(69)

where the top and bottom thermal block temperature timeseries measurements are $T_{i}^{\text{top}}(t)$ and $T_{i}^{\text{bottom}}(t)$, respectively, with $i = 1, ..., 6$ corresponding to azimuthal location (see figure 16), and the angled brackets represent averages in time. The fluid tem-
Figure 27: Sample temperature time series measurements from non-rotating convection in a 5 cm tank of water with 100 W heat power applied. a) and b) are temperature time series measurements from the top and, bottom thermistors, respectively, $T_{\text{top}}^i(t)$ and $T_{\text{bottom}}^i(t)$. c): The thermistor measurements from a) and b) are averaged to calculate a time series measurement of the mean fluid temperature, $T_{\text{fluid}}(t) = \frac{1}{12} \sum_{i=1}^{6} \left( T_{\text{top}}^i(t) + \sum_{i=1}^{6} T_{\text{bottom}}^i(t) \right)$. d): The difference between mean top and bottom temperatures gives the temperature drop, $\Delta T(t) = \frac{1}{6} \left( \sum_{i=1}^{6} T_{\text{bottom}}^i(t) - \sum_{i=1}^{6} T_{\text{top}}^i(t) \right)$. These measurements are then averaged in time for each convection case, and used to calculate fluid properties and diagnostic parameters.

Using this mean fluid temperature, the fluid properties for a given convection run are
calculated using equations 40 - 53. This permits a calculation of the Prandtl number, 

$$Pr = \frac{\nu}{\kappa}$$

for each case.

In order to measure the Ekman number, 

$$E = \frac{\nu}{(2\Omega L^2)}$$

and tank height \( L \) must also be measured. The rotation rate is measured by the servodrive, which reports a voltage drop to our acquisition system, as detailed above. This voltage drop is converted into a rotation rate via the manufacturer prescribed relation:

$$\Omega_{\text{RPM}} = \frac{(481.5564 V_{\text{servo}} - 12.517095)}{40}, \quad (70)$$

where \( \Omega_{\text{RPM}} \) is the rotation rate in rotations per minute, and \( V_{\text{servo}} \) is the voltage drop measured across the signal from the servodrive. The tank height, \( L \), is measured using a digital micrometer. These measurements permit a calculation of the Ekman number for each convection case.

The Rayleigh number, 

$$Ra = \frac{\alpha g \Delta TL^3}{\nu \kappa}$$

is further dependent on \( g \) and \( \Delta T \). Gravitational acceleration is assumed to be 9.81m/s\(^2\). The temperature drop is measured using the top and bottom thermistors:

$$\Delta T = \frac{1}{12} \left( \sum_{i=1}^{6} T_{i,\text{bottom}}(t) - \sum_{i=1}^{6} T_{i,\text{top}}(t) \right), \quad (71)$$

where, again, the angled brackets indicate averages in time. Figure 27d shows the measurement of \( \Delta T \) prior to time-averaging. Thus, we can now calculate the Rayleigh number for each convection run.

In order to calculate the Nusselt number, 

$$Nu = \frac{qL}{k\Delta T}$$

the heat flux \( q = P/A \), where \( P \) is heat power and \( A \) is the area through which that heat power flows, must
be measured. As mentioned above, the heat power is measured as the electrical power supplied to the heater. The area of the convection tank is calculated from the design drawings to be $A = 0.0314 \text{ m}^2$. These additional measurements allow us to calculate the fluid’s Nusselt number for each convection case.

These data, $Pr$, $E$, $Ra$, and $Nu$, make up the bulk of the experimental results presented in this document, and are reported in appendix B for all convection cases.

The temperature cross-covariance measurements presented in section 3.4 are calculated using the top and bottom thermistors measurements. Cross-covariances are calculated using temperature timeseries measurements of top and bottom thermistors at the same location in azimuth. This is done using the MATLAB function $xcov$. I use this program to calculate the cross-covariance of each pair as

$$C_{i}^{\text{top-bottom}}(m) = \sum_{t/dt=0}^{N-|m|-1} \left( T_{i}^{\text{top}}(t + m \, dt) - \langle T_{i}^{\text{top}}(t) \rangle \right) \left( T_{i}^{\text{bottom}}(t + m \, dt) - \langle T_{i}^{\text{bottom}}(t) \rangle \right),$$

(72)

where $t$ is the measurement time and $dt$ is the inverse of the data acquisition frequency such that $t/dt = 0, 1, ..., N$, where $N$ is the total number of data points in time, and $m = 1, ..., 2N - 1$ is the covariance lag. $C_i(m)$ is then normalized such that the auto-covariance at zero lag, $C_i^{\text{top-top}}(m = 0)$, is unity. The cross-covariances shown in figure 45 are the average zero-lag covariance of the six pairs:

$$T_{\text{cov}} = \sum_{i=1}^{6} C_{i}^{\text{top-bottom}}(m = 0),$$

(73)

for each convection case.

The central temperature variance measurements (figure 47) are calculated using
measurements from an internal thermistor whose tip is located near the center of the convection tank. These temperature time series are annotated $T_{\text{center}}(t)$. The variances reported in this document are calculated as

$$T_{\text{variance}}^{\text{center}} = \frac{1}{N} \sum_{t/dt}^N \left( T_{\text{center}}(t) - \langle T_{\text{center}}(t) \rangle \right)^2. \quad (74)$$

Essentially, then, the variance is the square of a standard deviation. The data shown in figure 47 are normalized by the temperature drop, $T_{\text{variance}}^{\text{center}}/\Delta T$. This calculation permits the comparison of temperature fluctuations in the fluid for each convection case.

2.8 Experimental Capabilities

It is RoMag’s broad range of accessible $Ra$, $E$, and $\Lambda = QE$ that make it a unique and valuable apparatus. For example, previous experimental devices have been able to access high $Ra$ (e.g., Glazier et al. (1999)) for non-rotating, non-magnetic convection, or $\Lambda \approx 1$ (e.g., Aurnou and Olson (2001)) for weakly supercritical convection, but never has this range of the three key ingredients of core flow been accessible in a single device. I developed the experimental device with this objective in mind. Although I do not present results from experiments in gallium in this document, I describe the experimental capabilities of RoMag below.

Figure 28 shows RoMag’s capability in accessing a broad range of $Ra$-space in liquid metal. Lower limits in Rayleigh number accessibility in a given tank and fluid are determined by our ability to measure small heat flux or temperature drop. Upper
Rayleigh-Benard Convection in Liquid Metals

Figure 28: Experimental accessibility of Rayleigh-space in liquid metal. For comparison, I include the most comprehensive previous studies of non-rotating, non-magnetic convection in low-$Pr$ fluids.

limits in the Rayleigh number will be determined by total heat power or total temperature drop. For the purposes of this discussion, we focus on the latter (above a $ΔT$ of about 40 °C, the average temperature of the fluid will begin to increase, and some experimental components may melt). RoMag, designed for rotating convection and magnetoconvection, is comparable to the most extreme parameters reached by experiments without these additional ingredients. When rotation is included, the contrast
between our capabilities and those from previous studies becomes more drastic.

Figures 29 and 30 show the range of Rayleigh and Ekman numbers accessible using RoMag. Lower limits on the Ekman number for a given tank size and fluid are determined by our maximum rotation rate. We fix our maximum rotation rate at 60 RPM, above which centrifugal perturbation of the effective gravity field within the tank may become important (Hart (2000)). In water, it is our range of accessibility
Figure 30: Experimental accessibility of Rayleigh-Ekman-space in gallium. For comparison, I include the space accessed by the most comprehensive previous rotating convection study in liquid metal, that by Rossby (1969) in mercury.

that differs from previous works. In gallium, both our range and the extremity of that range are far beyond what has been previously done.

Figure 31 shows the range of Rayleigh and Chandrasekhar numbers accessible using RoMag. Upper limits on $Q$ for a given tank size are determined by our maximum magnetic field strength, 0.13 T. Again, the range of space accessible to RoMag is
Figure 31: Experimental accessibility of Rayleigh-Chandrasekhar-space in gallium. For comparison, I include the space accessed by the most comprehensive previous magnetoconvection study in liquid metal, that by Cioni et al. (2000) in mercury.

Figure 32 shows the accessible range of Ekman-Chandrasekhar space in rotating magnetoconvection. RoMag is able to reach parameters values that have not been previously explored.
Figure 32: Experimental accessibility of Ekman-Chandrasekhar-space in gallium. The dashed line indicates where the Elsasser number is unity, \( \Lambda = 1 \). For comparison, I include the space accessed by the most comprehensive previous rotating magnetoconvection study in liquid metal, that by Aurnou and Olson (2001) in liquid gallium.

### 2.9 Laboratory-Numerical Collaboration

I have studied the problem of planetary core convection using RoMag in collaboration with numerical simulations. Two suites of simulations are included in the studies presented here.
The first is a suite of plane layer rotating convection simulations presented in section 3, which were conducted by Stephan Stellmach, of the University of Münster. Details of Stephan’s numerical method are given in appendix A.2. Essentially, the geometric setup of the plane layer simulations is nearly the same as in RoMag (except, e.g., that the simulations utilize periodic sidewall boundary conditions, whereas they are necessarily rigid in the experiment). Although these simulations are not capable of reaching the same extremes in convection parameters, they provide the distinct advantage of a wealth of diagnostic tools, which are often lacking in experimental studies.

The second is a suite of numerical dynamo models discussed in section 4.1. The models were carried out by fellow UCLA graduate student Krista Soderlund, and supplemented by an array of dynamo data from Ulrich Christensen, of the Max-Planck Institute for Solar System Research in Katlenburg-Lindau, Germany. These simulations are Boussinesq, convectively driven dynamo models in rotating spherical shells. Further model details are provided in appendix A.3.
3 Plane Layer Rotating Convection
3.1 Heat Transfer Background

Convective heat transfer has been studied in various capacities by researchers in many different areas of physical science and engineering. The efficiency with which fluid motions transport heat provide valuable information about the fluid motions themselves. Again, the Nusselt number characterizes this efficiency, as

\[ Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Heat Transfer}} = \frac{qL}{k\Delta T}, \tag{75} \]

where \( q \) is the heat flux, \( L \) is the container’s length scale, \( k \) is the fluid’s thermal conductivity, and \( \Delta T \) is the total temperature drop. A wealth of previous experimental studies of heat transfer exist for non-rotating, Rayleigh-Bénard convection. Early experiments seemed to corroborate the classical \( \alpha = 1/3 \) scaling (e.g., Dropkin and Globe (1959)). Later, Castaing et al. (1989) discovered the so-called ‘hard turbulence’ regime. For \( Ra \gtrsim 10^7 \), they observed a scaling with \( \alpha = 2/7 \). Scaling laws with \( \alpha \approx 2/7 \) have been found by several other studies since (e.g., Takeshita et al. (1996), Glazier et al. (1999)). At this writing, no explanation of this empirical scaling law has become generally accepted. The classical assumption that the temperature drop occurs in the boundary layers, and that \( Nu \propto \delta_k^{-1} \) have held as valid (Belmonte et al. (1994)). The breakdown of the classical \( \alpha = 1/3 \) scaling tells us then that the boundary layers are somehow communicating.

Experimental investigations of heat transfer in rapidly rotating convection have received much attention in recent years. Researchers have examined rotating convection in the laboratory and with numerical models in various ranges of parameter
space and with many different geometries. Several rotating plane layer experiments have found scaling laws in agreement with the non-rotating \( \alpha = 2/7 \) scaling (e.g., Julien et al. (1996b), Liu and Ecke (1997)). In these studies, the convective Rossby number, \( R_{oc} \), is fixed to a value that is less than unity.

Rapidly rotating convection in spherical shells has also been investigated experimentally and numerically (e.g., Sun and Schubert (1995), Sumita and Olson (2000)). Notably, several successful scaling laws for heat transfer and velocity information have been developed such that, following Kolmogorov’s second hypothesis of similarity (Kolmogorov (1941)), they are independent of molecular diffusivities (e.g., Christensen (2002), Christensen and Aubert (2006)). Instead, the scalings utilize the rotational timescale, \( \Omega^{-1} \) to generate new diffusivity-free parameters:

\[
Ra^*_Q = RaNuE^3Pr^{-2};
\]

\[
Nu^* = NuEPr^{-1}.
\]

Parameter surveys of heat transfer behavior seek scalings \( Nu^* \propto Ra^*_Q^\beta \), and have found \( \beta \approx 6/11 \) (e.g., Christensen (2002), Christensen and Aubert (2006)). The diffusivity-free scaling can be related to the standard \( Nu \propto Ra^\alpha \) scaling as:

\[
\alpha = \frac{\beta}{1 - \beta},
\]

and thus the aforementioned numerical studies find scalings with \( \alpha \approx 6/5 \) (Aurnou (2007)). Here, it should be mentioned, we consider heat transfer scalings for fixed \( E \), whereas in the aforementioned rotating convection experiments, which find \( \alpha \approx 2/7 \),
$Ro_e$ is fixed. That said, we see two very different empirical scalings: one that behaves as non-rotating convection; and another, which follows a much steeper scaling than the non-rotating behavior. Extrapolating these two laws to astrophysical or geophysical parameters yields predictions that disagree by many orders of magnitude.

We aim to clarify this discrepancy by surveying heat transfer behavior over a large range of parameter space. Can we find both the $\alpha \approx 6/5$ and $\alpha \approx 2/7$ scaling in a rotating plane layer? If so, where and why does the transition between the two occur? In order to answer these questions, we carry out laboratory and numerical experiments spanning

$$2 \times 10^3 < Ra < 6 \times 10^9 \quad (79)$$
$$10^{-6} \leq E \leq \infty \quad (80)$$
$$1 \leq Pr \leq 100. \quad (81)$$

### 3.2 Heat Transfer Results: The Transition

Experimental heat transfer behavior is shown in figure 33. Non-rotating convection in our laboratory experiments yields $Nu \sim Ra^{0.289\pm0.005}$ across more than five decades in $Ra$, in agreement with the previously obtained $Nu \propto Ra^{2/7}$ non-rotating scaling law (e.g., Glazier et al. (1999)). Different $Pr$ fluids yield similar non-rotating scalings with slightly different scaling prefactors (e.g., Verzicco and Camussi (1999), Schmalzl et al. (2002)), and this relatively weak effect is not considered here.

When rotation is included, the onset of convection is delayed by the stabilizing
Figure 33: Nusselt number, $Nu$, versus Rayleigh number, $Ra$, for a) laboratory experiments and b) numerical simulations. Laboratory experiments were carried out in cylinders with diameter to height ratios ranging from 6.25 to 1 using water ($Pr \approx 7$) and sucrose solution ($Pr \approx 10$). Included in a) are results from Rossby (1969) in water. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)). Solid black lines represent the non-rotating scaling law $Nu = 0.16Ra^{2/7}$. Dashed black lines represent the rapidly rotating scaling law $Nu = (Ra/Ra_c)^{6/5}$, where $Ra_c = 6E^{-4/3}$ from Chandrasekhar (1953).
effect of the Coriolis force (Chandrasekhar (1953)). Once convection begins, heat transfer exhibits a much steeper scaling, in agreement with the previously reported $\alpha \approx 6/5$ relationship (e.g., Christensen (2002), Christensen and Aubert (2006)). More specifically, heat transfer is adequately described by $Nu = (Ra/Ra_c)^{6/5}$ in this convective regime, where $Ra_c = 6E^{-4/3}$ is the critical Rayleigh number for the onset of convection (Chandrasekhar (1953)). However, due to the experimental limitations in accessing the rapidly rotating, strongly supercritical regime, this scaling is not yet well constrained.

For each Ekman number, our heat transfer data conform to the non-rotating scaling behavior beyond a given Rayleigh number. Thus, we observe two distinct convective heat transfer regimes: the rotationally controlled regime, $Nu \sim Ra^{6/5}$, and the non-rotating style regime, $Nu \sim Ra^{2/7}$. That rotation is relatively unimportant for heat transfer in the latter of these two regimes is indicative of a general lack of rotational control. We are thus interested in where and why this transition occurs.

We define the transition between these two regimes as the point of intersection between their respective scalings, $Nu = 0.16Ra^{2/7}$ and $Nu = (Ra/Ra_c)^{6/5}$. Equating the two, we solve for transitional Rayleigh and Nusselt numbers:

$$Ra_t = 1.7E^{-7/4}; \quad Nu_t = 0.16E^{-1/2}.$$  \hspace{1cm} (82)

Figure 34 shows $Nu$ normalized by the non-rotating scaling law versus $Ra$ normalized by the transitional Rayleigh number. When $Ra < Ra_t$ ($Nu < Nu_t$), convection is constrained by the influence of rotation, and heat transfer is less efficient than its non-
Figure 34: The transition from rotationally controlled to non-rotating heat transfer behavior for $E \leq 10^{-3}$ and $1 \leq Pr \leq 100$. On the horizontal axis, the Rayleigh number, $Ra$, is normalized by the transitional Rayleigh number, $Ra_t = 1.7E^{-7/4}$. Shown on the vertical axis is the Nusselt number, $Nu$, normalized by its non-rotating scaling, $Nu_{Non-Rotating} = 0.16Ra^{2/7}$. Symbols are the same as in figure 33, with experimental data shown as circles and stars, and numerical data shown as squares and diamonds. Numerical simulations are carried out by Stephan Stellmach.

rotating counterpart. When $Ra > Ra_t$ ($Nu > Nu_t$), heat transfer is not significantly affected by rotation and follows the non-rotating scaling. The rotating convection data overshoot the non-rotating scaling law near the regime transition. This slight enhancement of convective heat transfer by rotation has been observed previously, and is attributed to Ekman pumping effects (Julien et al. (1996b), Kunnen et al. (2006)). It should be noted that the observed heat transfer transition is not perfectly
sharp. If we define the transition as the point at which $Nu = Nu_{\text{Non-Rotating}}$, then we see that this transition is distributed across almost an order of magnitude of $Ra/Ra_t$ between $E = 10^{-3}$ and $E = 3 \times 10^{-6}$. In order to better resolve the transition location, lower $E$ convection studies are required.

It has long been argued that the influence of rotation on turbulent convection dynamics is well characterized by the convective Rossby number, which predicts a transition between rotationally controlled convection and non-rotating style convection when $Ro_c = \sqrt{RaE^2 Pr^{-1}} \approx 1$ (e.g., Aurnou et al. (2007)). The convective Rossby number thus predicts a transitional Rayleigh number that scales as $E^{-2} Pr$, compared to the $E^{-7/4}$ scaling derived from our boundary layer arguments. These two scalings, when extrapolated to planetary settings, yield drastically different predictions of the importance of rotation. Using a typical estimate for Earth’s core of $E = 10^{-15}$, for example, the two scalings predict transitional Rayleigh numbers that differ by roughly four orders of magnitude.

Figure 35 shows experimental heat transfer data plotted with $Ro_c$. Should this global force balance argument control the importance of rotation in convective heat transfer, we would expect the transitions to occur when $Ro_c \approx 1$. The heat transfer transitions observed in the data (figure 35) are not adequately explained by the global force balance, $Ro_c$, but, instead, are well described by our empirical transition scaling. Furthermore, the force balance argument predicts a $Pr$-dependent transition, and no such dependence is observed.
Figure 35: Nusselt number versus the convective Rossby number for laboratory experiments in water (circles) and from numerical simulations carried out by Stephan Stellmach (squares) with $Pr = 7$ (King et al. (2009)). The convective Rossby number, $Ro_c$, characterizes the ratio of buoyancy forcing to the Coriolis force. Force balance arguments predict that rotation will dominate the system when $Ro_c < 1$. However, observed heat transfer regime transitions more closely follow the boundary layer controlled transitional Nusselt number, $Nu_t = 0.18E^{-1/2}$ (dashed line).

Figure 36 shows heat transfer data from an array of previous rotating convection studies that is re-scaled by the transitional Rayleigh number. We observe that this transition scaling collapses the disparate results from previous studies: those finding the rotationally controlled $Nu \sim Ra^{6/5}$ heat transfer behavior typically have $Ra < Ra_t$ (Christensen (2002), Aubert (2005), Christensen and Aubert (2006)); while those studies yielding the non-rotating style $Nu \sim Ra^{2/7}$ scaling typically have
Figure 36: Convective regime transitions from previous studies. The Nusselt number is normalized by its non-rotating counterpart $Nu_{\text{Non-Rotating}} = A Ra^{2/7}$, with $A = 0.16$ and 0.33 for no-slip and stress free boundaries, respectively. The Rayleigh number is normalized by its transitional counterpart, $Ra_t = B E^{-7/4}$, with $B = 1.7$ and 5.3 for no-slip and stress free boundaries, respectively. Spherical studies are represented by circles, and plane layer studies by all other symbols. Studies with stress-free boundary conditions are in blue. We find that the transitional Rayleigh number describes heat transfer scaling transitions among a diverse array of previous studies.

$Ra > Ra_t$ (Julien et al. (1996b), Liu and Ecke (1997), Julien et al. (1996a)), despite having $Ro_c < 1$. Note that the non-rotating scaling law has a higher prefactor for stress-free boundary conditions than no slip conditions in the plane layer (section 3.5): $Nu_{\text{Non-Rotating}} = A Ra^{2/7}$, with $A = 0.16$ and 0.33 for no-slip and stress
free boundaries, respectively. This leads to different prefactors for the transitional Rayleigh number for the different boundary conditions: $Ra_t = BE^{-7/4}$, with $B = 1.7$ and 5.3 for no-slip and stress free boundaries, respectively (see section 3.5 for stress free discussion). However, the stress-free spherical study from Christensen (2002) appears to transition to a slightly less efficient non-rotating law than its no-slip counterparts. This suppression of heat transfer may be due to the strong zonal winds that can develop in stress-free spherical shells, which can act to inhibit convective heat transfer (Aurnou et al. (2008)). See section 4.1 for further discussion on application to spherical dynamos.

3.3 The Role of Boundary Layers

We hypothesize that the effects of rotation dominate convection dynamics when the Ekman layer is thinner than the thermal boundary layer. The thermal boundary layer is the thin region of fluid through which heat is diffused from the boundary to the interior fluid, as was discussed in section 1.2.3. The Ekman boundary layer is the region wherein viscosity becomes important in enforcing boundary conditions in rapidly rotating containers. Greenspan (1968) shows that the Ekman layer plays a prominent role in controlling flows that depart from quiescent solid body rotation. The Ekman layer can be thought of as the region over which a container’s rotation is communicated to the interior fluid. We can solve for the Ekman layer thickness, $\delta_E$. 
by balancing the viscous and Coriolis terms in equation (7) on page 15:

\[ 2\Omega \hat{z} \times \mathbf{u} \sim \nu \nabla^2 \mathbf{u} \implies 2\Omega U \sim \nu \delta_E^{-2} \implies \frac{\delta_E}{L} \sim \left( \frac{\nu}{2\Omega L^2} \right)^{1/2}. \]  

(83)

Thus, our two important boundary layers are the thermal and Ekman layers, which have respective thicknesses

\[ \delta_\kappa \sim Nu^{-1} \quad \text{and} \quad \delta_E \sim E^{1/2}. \]  

(84)

The thermal boundary layer corresponds to the location from which thermal plumes depart, and the fluid interior (the remaining volume) is considered well mixed (e.g., Spiegel (1971)). We hypothesize that when the thermal boundary layer is thinner than the Ekman layer, \( \delta_\kappa < \delta_E \), the uppermost part of the Ekman layer is mixed with the bulk. This mixing may act to truncate the influence of the Ekman layer, and therefore rotation, on the interior fluid dynamics (e.g., Hignett et al. (1981), Read (2001)). In contrast, when \( \delta_E < \delta_\kappa \) the Ekman layer survives in full, and we anticipate interior dynamics which are controlled by rotation. The transition is then predicted to occur where the boundary layers cross; i.e. when \( \delta_E = \delta_\kappa \). The Nusselt number at which the transition should occur is then

\[ Nu_t \sim E^{-1/2}. \]  

(85)

The Nusselt number can be related to the Rayleigh number using the heat transfer scaling law \( Nu \sim Ra^{2/7} \). This allows us to solve for the Rayleigh number at which the boundary layers are predicted to cross:

\[ Ra_t = E^{-7/4}, \]  

(86)
called the transitional Rayleigh number. These theoretically derived transitional Nusselt and Rayleigh numbers agree with our empirical finding (figures 34, 35, and 36).

Figure 37: Vertical profiles of a) mean temperature (non-dimensionalized by the total temperature drop, $\Delta T$), b) temperature variance, and c) RMS velocity (non-dimensionalized by $\kappa/L$) for $E = 10^{-4}$, $Pr = 7$, and $1.9 \times 10^6 \leq Ra \leq 2.1 \times 10^8$ from numerical experiments carried out by Stephan Stellmach (King et al. (2009)). The thermal boundary layer thickness is defined as the height of the peak value of the temperature variance. The Ekman layer thickness is defined as the height of the peak values of the RMS velocity (e.g., Belmonte et al. (1994)).

To test our hypothesis further, we measure the thicknesses of the two boundary layers in the numerical simulations. Figure 37 shows vertical profiles of mean temperature, temperature variance, and velocity. Following Belmonte et al. (1994), we define the Ekman boundary layer thickness $\delta_E$ as the vertical position of the maximum value of the root mean square velocity, and the thermal boundary layer thickness $\delta_\kappa$ as the vertical position of the maximum value of the temperature variance. Isothermalization of the interior fluid occurs for $Ra/Ra_t > 1$, and permits the formation of
a well defined thermal boundary layer. The thickness of this boundary layer is well
described by the height of the peak value of the temperature variance (defined as the
square of the standard deviation of temperature in time), physically corresponding
to the location of the development and departure of thermal plumes before they are
mixed by the turbulent interior (Tilgner et al. (1993), Takeshita et al. (1996)) When
$Ra/Ra_t > 1$, thermal boundary layers are well-defined and the interior fluid is nearly
isothermal.

Figure 38 shows both $Nu$ and $L/2\delta_\kappa$ from the $E = 10^{-4}$, $Pr = 7$ numerical ex-
periment. The agreement between heat transfer and boundary layer thickness mea-
urements show that the boundary layer scaling of the Nusselt number, $Nu = L/2\delta_\kappa$,
is a valid approximation (see equation (29) on page 22).

Figure 39 shows measurements of the thermal and Ekman boundary layer thick-
esses from the $E = 10^{-4}$, $Pr = 7$ numerical experiments. When $Ra < Ra_t$, we find
$\delta_E < \delta_\kappa$. Furthermore, when $Ra > Ra_t$, we find that $\delta_\kappa < \delta_E$. Thus, as predicted by
the boundary layer control hypothesis, the heat transfer transitions are well described
by the competing boundary layers.

To test this idea more generally, we can compare the boundary layer competition
with the heat transfer data across a broader range of parameters. Figure 40 shows
the ratio of the boundary layer thicknesses as well as the ratio between the Nusselt
number and its non-rotating scaling for $10^{-3} \leq E \leq 10^{-5}$ and $1 \leq Pr \leq 10$. Our
hypothesis predicts that these two ratios will reach unity at the same Rayleigh number
Figure 38: Nusselt number versus Rayleigh number for $E = 10^{-4}$, $Pr = 7$ numerical experiments. Also shown are thermal boundary layer measurements $L/2\delta$, where $\delta$ is the thermal boundary layer thickness, defined as the height of the peak value of the temperature variance (Tilgner et al. (1993)). Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).

for given Ekman and Prandtl numbers. That is, the two ratios should intersect near unity. Figure 40 illustrates the agreement between the boundary layer hypothesis prediction and empirical heat transfer behavior. When $\delta/\delta_E > 1$, heat transfer is suppressed by the influence of rotation. Conversely, when $\delta/\delta_E < 1$, heat transfer is not strongly influenced by rotation. Also notable is the relative unimportance of varying $Pr$, which further supports our ideas of boundary layer control.

In figure 39, we see that heat transfer scaling transitions occur when the bound-
Figure 39: The non-dimensional thicknesses of the competing boundary layers are shown as the Rayleigh number is varied for $E = 10^{-4}$ and $Pr = 7$ from numerical experiments. The dynamical transition at $Ra = Ra_t$ occurs when the relative thicknesses of the competing boundary layers are approximately equal, $\delta_\kappa = \delta_E$. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).

It is important to note here the difference between the boundary layer thickness ratio and the transition prediction $Ra/Ra_t$. The transitional Rayleigh number is a parameterization of where we expect the boundary layers to cross based on how the thickness of each scales ($\delta_E \sim E^{-1/2}$, $\delta_\kappa \sim Nu^{-1}$). In order to scale the thermal boundary layer in terms of the control parameter $Ra$, we must invoke the heat transfer scaling law $Nu \sim Ra^{2/7}$. Thus, we are assuming here that the ther-
Figure 40: Boundary layer control of heat transfer transitions for \(10^{-5} \leq E \leq 10^{-3}\) and \(1 \leq Pr \leq 100\) from numerical experiments. Circles depict \(Nu\) normalized by the non-rotating scaling law, \(Nu_{\text{Non-Rotating}} = 0.16 Ra^{2/7}\). Stars represent the ratio of the thermal boundary layer thickness to the Ekman boundary layer thickness. The Ekman layer thickness is defined as the mean height of the peak value of the RMS velocity (Belmonte et al. (1994)). Three different \(Pr\) values are shown: \(Pr = 1\), dotted lines; \(Pr = 7\), dashed lines; \(Pr = 100\), solid lines. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).

Imperfect boundary layer scales as \(\delta_k \sim Ra^{-2/7}\) in the supertransitional regime. Imperfect boundary layer thickness scalings may make \(Ra_t\) an imperfect transition scaling.
Figure 40 shows the heat transfer data normalized by the non-rotating scaling law, $Nu/Nu_{\text{Non-Rotating}}$ (circles), and the boundary layer thickness ratios (stars) versus $Ra$ for three Ekman numbers from the numerical simulations. We observe that the heat transfer data cross unity approximately where the boundary layer ratio crosses unity. This supports our hypothesis, which predicts that heat transfer will conform to the non-rotating behavior when the thermal boundary layer becomes thinner than the Ekman boundary layer. Also, note that the first order behavior of these quantities is independent of $Pr$. Figure 40 illustrates that the boundary layers themselves adequately describe the transition in heat transfer behavior. In order to improve our scaling of the transition location, we need to understand better how the boundary layers scale with our control parameters.
3.4 Exploring the Transition: the breakdown of quasigeostrophic structures

The transition between these two different heat transfer regimes reveals a change in convective flow behavior. I will argue that the transition from rotationally constrained to non-rotating style heat transfer behavior is linked to the breakdown of the large-scale, coherent flow structures typical of quasigeostrophic convection. In the rotationally controlled regime, we see evidence of Taylor columns: tall, thin vortical flow structures that constitute so called quasi-two-dimensional convection. In the non-rotating style heat transfer regime, we find evidence for strongly three-dimensional turbulence.

Figure 41 shows a comparison of flow structures from both laboratory experiments and numerical simulations in each regime. Laboratory visualizations (top row) are accomplished by shining a vertical sheet of laser light through a light solution of Kalliroscope in water. Numerical visualizations (bottom row) display iso-surfaces of vertical velocity. Case \(a\) has \(Ra = 1.7 \times 10^9\) and \(E = 3 \times 10^{-5}\), and so \(Ra/Ra_t = 0.22\). Case \(b\) has \(Ra = 1.7 \times 10^9\) and \(E = 3 \times 10^{-6}\), and so \(Ra/Ra_t = 12.2\). Case \(c\) has \(Ra = 7 \times 10^6\) and \(E = 10^{-4}\), and so \(Ra/Ra_t = 0.41\). Case \(d\) has \(Ra = 2 \times 10^8\) and \(E = 10^{-4}\), and so \(Ra/Ra_t = 12.8\). All four have \(Pr = 7\). When \(Ra < Ra_t\) (left column), which we shall call subtransitional convection, we see large-scale, coherent flow structures which are aligned with the rotation axis. These quasi-two-dimensional vortical flow structures are often referred to as Taylor columns. Movies of convective
Figure 41: Visualizations of flow from laboratory experiments (first row) and numerical simulations (second row). Laboratory visualizations are accomplished by shining a vertical sheet of laser light through a light solution of Kalliroscope in water. Numerical visualizations display iso-surfaces of vertical velocity, where gold (blue) surfaces represent upward (downward) flow. Case a) has $Ra = 1.7 \times 10^9$ and $E = 3 \times 10^{-5}$. Case b) has $Ra = 1.7 \times 10^9$ and $E = 3 \times 10^{-6}$. Case c) has $Ra = 7 \times 10^6$ and $E = 10^{-4}$. Case d) has $Ra = 2 \times 10^8$ and $E = 10^{-4}$. All four have $Pr = 7$. Cases on the left then have $Ra < Ra_t$ and cases on the right have $Ra > Ra_t$. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).
flow from both laboratory and numerical experiments reveal that the Taylor columns tend to wander randomly about the container. In the supertransitional regime ($Ra > Ra_t$, right column), convection manifests as strongly three-dimensional flow. Here, it is more difficult to identify any preferred organization of flow structure, similar to traditional unconstrained turbulence. Notably, the supertransitional cases $b$ and $d$ have $Ro < 1$, yet display strongly three-dimensional convective structures.
Figure 43: Temperature correlations from vertical thermistor pairs in non-rotating convection. Cross-covariance of vertical temperature is defined in equation 72 on page 83.

Experimentally, we are able to see evidence of Taylor columns. We make temperature measurements in the solid top and bottom boundaries with twelve thermistor probes distributed such that they form six vertically aligned pairs. As an indirect probe of flow structures, we can look at correlations in the temperature time series on each pair. We expect subtransition convection, with its container-sized vertical columns, to exhibit strong thermal correlations as the columns drift around the container and past each of the six vertically aligned thermistor probes. Figure 42 shows
the temperature correlations across the six pairs for subtransitional convection, and figure 43 shows the temperature correlations for non-rotating convection. We observe strong negative correlations in the thermal signature of subtransitional convection, and find more incoherent thermal signatures in the non-rotating case.

That the subtransitional temperature signals are vertically anticorrelated is perhaps counterintuitive. It is important here to distinguish between temperature and heat. Taylor columns are composed of either positive or negative vertical velocity. Upward flowing Taylor columns will transport warm material from bottom boundary to top, and downward flowing columns will bring cold material from top to bottom. Taylor columns are then either anomalously hot or cold. If we could measure temperature signals on vertical pairs of thermistors within the fluid, we would then expect positive correlations.

However, we measure our temperature times series within the boundary itself. This difference is important in producing the anticorrelated thermal signals. Convective heat transfer can be parameterized locally as $uT'$, where $T'$ is the temperature anomaly (e.g., Hathaway (1984)) Both flavors of Taylor columns transport heat upward. This is accomplished by removing heat from the bottom boundary, and depositing it at the top boundary. A vertical conduit of heat, which we shall call a chimney, will simultaneously cool the bottom boundary and warm the top boundary. Within the boundaries themselves, then, we should see a negatively correlated temperature time signal.
To test this idea, I’ve developed a one-dimensional toy model of vertical heat transfer in the experiment. The model, which is described in detail in appendix A.1, consists of a ‘convection tank’ sandwiched between two thermal blocks. The temperature is fixed above the top thermal block, and a heat flux is fixed below the bottom thermal block. I treat the drifting Taylor columns as a chimney of fluctuating strength. That is, the ‘convection tank’ is modeled as a region of temporally variant, spatially invariant heat flux. The time series used for the model fluid’s Nusselt number is adopted from experimental Nusselt number time series. Dimensional quantities are used, which are accurate to material properties and experimental settings. I step the model through time, and calculate the vertical temperature structure of the model at each time step. I record synthetic temperature probe time series within the model’s top and bottom boundaries, and determine the correlation coefficient between the two, as in the actual experiment.

Figure 44 shows the results of the model’s temperature correlation, which qualitatively agree with that measured experimentally. This illustrates that the subtransitional experimental measurements are well described by wandering Taylor columns, which act as chimneys pumping heat upward. Importantly, it is heat transfer, not just temperature, that is aligned in the vertical.

It should be mentioned that there exists an alternative hypothesis to explain vertical correlations in heat transfer. If a plume leaves the bottom boundary and is able to cross the fluid layer intact, it will collide with the upper boundary, depositing
Figure 44: Vertical temperature cross-covariance versus time lag from a simple one-dimensional, time dependent heat transfer model. Qualitative agreement between the model and laboratory results (figure 42) illustrate that strong anticorrelations in the subtransitional regime can be described by traveling heat pipes. As heat transfer is assumed to be perfectly vertical, the correlation is perfect (-1) at zero time lag.

If the plume travels purely vertically across the layer, that heat transport will be vertically aligned. This mechanism fails to explain the vertical correlations we see, however, since the time lag $k$ of the peak correlation coefficient in this counterexample would correspond to the time it takes the plume to traverse the
Figure 45: Vertical temperature correlation coefficient (at time lag $k=0$) as a function of $Ra/Ra_t$ from laboratory experiments. Vertical temperature covariance is defined in equation 73 on page 83. Error bars represent the correlation coefficient variance between the six thermistor pairs for each experiment.

In our subtransitional correlation measurements, the peak time lag is always much shorter than the free-fall timescale. Thus, the efficient communication between the boundaries occurs much faster than would it take a plume to deliver heat.

Figure 45 shows how the vertical temperature correlations vary across our boundary layer controlled transition space. Strong anticorrelations are observed in the
subtransitional regime, indicative of the importance of the Taylor column chimney mechanism for heat transfer in rotationally controlled convection. That vertical temperature correlations are no longer anti-correlated in the supertransitional regime indicates that there may not be any strong direct communication between top and bottom boundaries by chimney convection structures. The observation of some significant positive cross-covariance in the supertransitional regime may be due to the existence of large-scale circulation patterns, which have been observed in previous studies of slowly rotating and non-rotating convection (e.g., Hart et al. (2002), Brown and Ahlers (2007)), and is observed visually in a limited number of cases in this study. This supports our working hypothesis that the transition is linked to the breakdown of the tall, thin convection structures. I conjecture that once the thermal boundary layer becomes thinner than the Ekman boundary layer, we should see convection dynamics similar to the non-rotating case, where three-dimensional, chaotic turbulence lead to a well-mixed interior.

Figure 37a on page 103 shows temperature profiles from several numerical cases. Here, we can see that the interior temperature gradient is decreasing as $Ra/Ra_t$ increases. To demonstrate more clearly the relationship between isothermalization of the convecting interior and the boundary layers, we can compare these temperature profiles with the measured boundary layer thicknesses.

To investigate this idea of enhanced mixing further, we can examine how vertical temperature profiles change across the regime transition. Figure 46 shows the mean
Figure 46: Normalized temperature gradients at the mid-plane versus the boundary layer thickness ratio for numerical simulations with $10^{-3} \leq E \leq 10^{-5}$ and $1 \leq Pr \leq 10$. The mid-plane temperature gradient is defined as the temperature lapse rate, $\partial T/\partial z$ at $z = L/2$ normalized by it’s conductive value, $\Delta T/L$. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).

We observe that the temperature gradient is close to zero when $\delta_\kappa < \delta_E$. Thus, as the thermal boundary layer becomes
thinner than the Ekman layer, the interior fluid becomes increasingly isothermal, illustrating the increasing efficiency of three-dimensional mixing. Interestingly, the lapse rate for the $Pr = 1$ case decreases more slowly beyond the boundary layer transition than for higher $Pr$. The slight flattening of the lapse rate of this case near the boundary layer crossing may explain the results from Julien et al. (1996b), who claim they see a finite internal temperature gradient at asymptotically high $Ra$ in a $Pr = 1$ fluid. This finding is likely due to their proximity to the boundary layer transition (see figure 36), where we see this feature of the lapse rate for our $Pr = 1$ case. This ‘knee’ in the lapse rate may also indicate a possible change in dynamics for lower $Pr$ fluids.

Another way to probe the properties of turbulent mixing in the interior is through temperature time series statistics. If we measure temperature at a single point within the fluid over time, we may expect different statistical signatures in each regime. As a Taylor column passes by the thermistor, we expect the probe to measure much warmer or cooler than average temperatures. In the supertransitional regime, we expect strong mixing to limit the amplitude of deviations from average temperature in the interior (e.g., Takeshita et al. (1996)). Figure 47 shows temperature variance versus $Ra/Ra_t$ from the laboratory experiments with $E \approx 10^{-4}$. The relative temperature variance increases until $Ra \approx Ra_t$, which indicates the growing strength of the Taylor columns as convective supercriticality increases. In the supertransitional regime, the central temperature variance decreases as is the interior fluid becomes more well
Figure 47: Temporal temperature variance (normalized by $\Delta T$) at the tank's center versus $Ra/Ra_t$ for experimental runs with $E \approx 10^{-4}$. Central temperature variance is defined in equation 74 on page 84. The gap in data between $Ra/Ra_t \approx 4$ and $Ra/Ra_t \approx 15$ is due to a change in experimental tank size.

mixed. Here, the lack of extreme departures from mean temperature at the midplane indicates that no strong temperature anomalies survive in the interior (Takeshita et al. (1996)).
3.5 A Stress Free Transition

Figure 48: Iso-surfaces of vertical velocity for a) $Ra = 7 \times 10^6$ and b) $Ra = 2 \times 10^8$, $E = 1 \times 10^{-4}$, and $Pr = 7$, from numerical simulations with stress-free boundary conditions. Case a) has $Ra/Ra_t = 0.13$. Case b) has $Ra/Ra_t = 3.8$ and $Ro_c = 0.5$. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).

We observe similar behavior in numerical experiments with stress-free boundaries, which may at first seem at odds with the boundary layer control hypothesis (Schmitz and Tilgner (2009)). Theoretical work by Hide (1964), however, shows that the thermal wind balance, the phenomenon responsible for ‘spinning up’ plumes to form Taylor columns in systems with stress free boundaries, becomes important at a distance from the boundary that scales as $E^{1/2}$. This free-slip version of the Ekman layer, called the thermal Ekman layer, is the vertical distance from the boundary over
which rotation is able to respond to density perturbations and influence plume formation. Julien et al. (1996b) show further that the boundary layer response to lateral variations of temperature is identical for no-slip and stress free boundary conditions. While the no-slip condition requires that flow, $u$, goes to zero at the boundaries, the stress free condition requires that vertical shear, $\partial u / \partial z$, goes to zero at the boundaries. This condition on the shear must be made to match interior flow similarly to the velocity in the no slip case, and over the same $E^{1/2}$ length scale, as long as thermal heterogeneities exist somewhere within the boundary layer (and they must for convection to occur). We further hypothesize, then, that the boundary layer control arguments put forth should apply to rotating convection systems with stress free boundaries.

I analyze a complementary suite of numerical rotating convection experiments with free-slip boundaries conducted by Stephan Stellmach (King et al. (2009)). Visualizations of flow structures from these simulations are shown in figure 48 as iso-surfaces of vertical velocity for a) $Ra = 7 \times 10^6$ and b) $Ra = 2 \times 10^8$, $E = 1 \times 10^{-4}$, and $Pr = 7$. Case a) has $Ra / Ra_t = 0.13$ and displays large-scale, coherent, axially aligned velocity structures typical of rotating convection. Case b) has $Ra / Ra_t = 3.8$ and shows turbulent, three-dimensional convective structures typical of non-rotating convection, despite having $Ro_c = 0.5$. As in the no-slip case, we again observe a breakdown of Taylor columns as $Ra$ crosses $Ra_t$, which would not be predicted by the convective Rossby number, $Ro_c$. 

122
Figure 49: The Nusselt number as a function of the Rayleigh number for convection with free-slip boundaries, and $10^{-6} \leq E \leq \infty$, and $1 \leq Pr \leq 100$. Non-rotating heat transfer is well described by a $Nu = 0.33Ra^{2/7}$ scaling. Rotating convection follows $Nu = (Ra/Ra_c)^{6/5}$ for $Ra/Ra_c < 1$. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).

Heat transfer data from the free-slip simulations are shown in figure 49. Non-rotating heat transfer is well described by $Nu = 0.33Ra^{2/7}$. Rotating convection is again adequately described by $Ra = (Ra/Ra_c)^{6/5}$, where $Ra_c$ is the critical Rayleigh number for the onset of convection. We use $Ra_c = 9E^{-4/3}$ for free-slip boundaries (Chandrasekhar (1953)). For sufficiently large $Ra$, rotating convective heat transfer switches to the non-rotating scaling. As before, we define the transitional Nusselt
Figure 50: Heat transfer scaling transition in rotating convection with free-slip boundaries. The Nusselt number is normalized by the non-rotating scaling law, $Nu = 0.33Ra^{2/7}$. The Rayleigh number, $Ra$ is normalized by the stress free transitional Rayleigh number, $Ra_t = 5.3E^{-7/4}$. When $Ra/Ra_t > 1$, rotating heat transfer conforms to the non-rotating scaling. Numerical experiments are carried out by Stephan Stellmach (King et al. (2009)).

and Rayleigh numbers, $Nu_t$ and $Ra_t$, as the point of intersection between the two scalings, yielding $Nu_t = 0.5E^{-1/2}$ and $Ra_t = 5.3E^{-7/4}$. Figure 50 shows that, as for the no-slip experiments, the transition in heat transfer behavior is well described by the transitional Rayleigh number, $Ra_t$.

This transition scaling agrees with the boundary layer controlled scaling derived
in section 3.3. We are continuing to investigate the dynamics of the thermal Ekman layer in this context. We expect that we can approximate its thickness using \( \frac{\partial u}{\partial z} \) in the same way \( u \) is used for the no-slip case. That the boundary layer controlled transition pertains to free-slip convection, however, suggests that these results may be applied to convection systems that are not contained by rigid boundaries, such as stars and gas planets.

3.6 Conclusions and Further Considerations

Results from plane layer rotating convection experiments in moderate Prandtl number fluids are shown. Two regimes are observed: a rapidly rotating regime (or subtransitional regime); and a weakly rotating regime (or supertransitional regime). Convection in the rapidly rotating regime follows a \( Nu = (Ra/Ra_c)^{6/5} \) heat transfer scaling law, while supertransitional convection obeys a non-rotating, \( Nu \sim Ra^{2/7} \) heat transfer scaling law. That the transition between these two regimes is argued to be linked to the ratio between the thicknesses of the thermal and Ekman boundary layers. A predictive scaling of where the regime transition is developed: where \( Ra = Ra_c \sim E^{-7/4} \). Furthermore, the heat transfer regime transition is connected with transitions from large scale columnar flows to chaotic, three-dimensional convection.

The heat transfer behavior observed in the experiments is also observed in simulations with stress-free boundaries, in which the effects of an Ekman layer are expected
to be minimal. This may be explained by the fact that a ratio of boundary layer thicknesses is similar to a ratio of physical timescales. If a plume forms and departs from the thermal boundary layer on a timescale that is short compared to the rotation timescale, it won’t likely be affected strongly by rotation. Conversely, if the plume takes many rotations to develop, it is more likely to be spun-up by the Coriolis force, permitting an important influence of rotation on the resulting convection. Let us allow a thermal plume to form on a viscous timescale, \( \tau_{\text{plume}} \sim \frac{\delta^2}{\kappa} \). The transition should then occur when the plume formation timescale is roughly equal to the timescale of rotation, \( \tau_{\Omega} \sim \Omega^{-1} \). Thus, the regime boundary should occur where

\[
\frac{\tau_{\Omega}}{\tau_{\text{plume}}} \sim \frac{\nu}{\delta^2 \Omega} \sim Nu^2 E \sim 1,
\]

using equation (84). So the timescale argument also predicts a transitional Nusselt number \( Nu_t \sim E^{-1/2} \). However, it has not yet been shown that the viscous timescale accurately describes plume formation. We are currently working to analyze time series statistics in and near the thermal boundary layer to determine the validity of this argument. Furthermore, should this scaling argument hold, it may change for low \( Pr \). In low \( Pr \) fluids, it is possible that plume formation would be controlled not by viscous diffusivity, but rather by thermal diffusivity. Thus, in liquid metal for example, we may expect to find \( Nu_t \sim (E/Pr)^{-1/2} \).

I have argued that the breakdown of Taylor columns at \( Ra/Ra_t \approx 1 \) leads to an increased effectiveness of thermal mixing. I discuss this idea more in the following discussion of dynamo results, but I would like to propose another method for analyzing
future plane-layer work here. A more complete definition of thermal mixing may be found in instantaneous spatial temperature variances. At any given time, along any given line (horizontal versus vertical), we can find the second moment of temperature in the numerical simulations. The second moment, or temperature variance, tells us how isothermal is the convecting fluid along this line. We can do this for several different lines, for all the different cases done. We can then define horizontal and vertical mixing quantities, $M$:

$$M_{x_y}^{-1} = \sum_{x,y} \int_z \left( \frac{\int_z T(x, y, z) dz}{Z} - T(x, y, z) \right)^2 dz,$$

$$M_z^{-1} = \sum_{z} \int_y \int_x \left( \frac{\int_y \int_x T(x, y, z) dxdy}{XY} - T(x, y, z) \right)^2 dxdy,$$

respectively, where $Z$ is the height of the volume considered, and $X$ and $Y$ are the horizontal lengths of the volume considered. We expect isotropic mixing to produce roughly equal mixing quantities in each direction. In contrast, if subtransitional convection is not well mixed in the horizontal direction, we expect higher temperature variances in that direction. This is hinted at by the experimental temporal temperature variance showed in figure 47, but the richer spatial information contained in $M_{x_y}$ and $M_z$ is only available in the numerical simulations. The efficiency and thermal mixing and its anisotropic nature are important in core dynamics, and are discussed further in sections 4.2.1 and 5.3.
4 Dynamo Model Analysis
4.1 Heat Transfer in Spherical Dynamo Models

4.1.1 Introduction

I hypothesize that the idea of boundary layer controlled heat transfer applies to planetary dynamo models. Two main reasons why the plane layer results may fail to apply to the dynamo models are: the added complexity of a spherical geometry; and the additional dynamic influence of the magnetic field via Lorentz forces.

Rotating convection in spherical shells is capable of driving strong zonal flows (e.g., Christensen (2002), Heimpel et al. (2005)), which are not observed in plane layer convection. Strong shear in these zonal flows can influence the mechanics of heat transfer from inner to outer boundary (Aurnou et al. (2008)). Despite this possible complication, Aurnou (2007) shows that heat transfer behavior does not strongly differ between these two configurations. Figure 36 also suggests that the plane layer study may, in fact, apply to spherical geometries, since three of the datasets shown are from spherical studies (Christensen (2002), Aubert (2005), Christensen and Aubert (2006)).

The presence of magnetic fields may also call into question the application of the plane layer, hydrodynamic results to dynamo models. For example, the influence of the Lorentz force may act to constrain fluid motions further and inhibit heat transfer (Aurnou and Olson (2001), Cioni et al. (2000)). This effect may not be critically important as it has been shown that time-dependent flows in rotating systems (of which convection is one) are not strongly influenced by the presence of magnetic
Figure 51: The theoretical Ekman-Hartmann layer thickness normalized by that of the Ekman layer, $\delta_{EH}/\delta_E \equiv \left[\left(\frac{\Lambda}{2}\right)^2 + 1\right]^{1/2} / \left(\frac{\Lambda}{2}\right)^{1/2}$, versus the Elsasser number, $\Lambda$ (Gilman and Benton (1968)). The Ekman-Hartmann layer thickness only differs significantly from the Ekman layer thickness in the presence of strong magnetic fields.

fields (Jault (2008)). Also, in contrast with convection under the influence of strong, imposed magnetic fields, heat transfer behavior in self-consistent dynamo models is typically found not to differ significantly from that in comparable cases without magnetic fields (see figure 36).

Another complication is introduced by the fact that the Ekman layer is no longer an adequate representation of the viscous boundary layer for flows in the presence of magnetic fields, which is now instead an Ekman-Hartmann layer. The Ekman-Hartmann layer balances the viscous force at the bounding surfaces with both Coriolis and Lorentz forces, instead of the Coriolis force alone as in the case of the Ekman layer. The thickness of the Ekman-Hartmann layer scales differently than the Ekman layer and therefore may render the development of the transitional Rayleigh number,
\( Ra_t \sim E^{-7/4} \), inappropriate. The Ekman-Hartmann layer scales as (Gilman and Benton (1968), Debnath (1973))

\[
\delta_{EH} \sim \left[ \left\{ \left( \frac{\Lambda}{2} \right)^2 + 1 \right\}^{1/2} - \frac{\Lambda}{2} \right]^{1/2} E^{1/2},
\]

(90)

where \( \Lambda \) is the Elsasser number. The Elsasser number characterizes the ratio of the strength of the Lorentz force to that of the Coriolis force, and is defined in table 1 on page 7. Figure 51 shows the ratio of the theoretical Ekman-Hartmann layer and Ekman layer thicknesses versus \( \Lambda \). We argue that the difference between the two thickness scalings should only become important for inordinately strong (\( \Lambda > 10 \)) magnetic field generation. For example, in order for the Ekman-Hartmann layer to differ in thickness from the Ekman layer by more than a factor of two (\( \delta_{EH}/\delta_E < 1/2 \)), the magnetic field must be quite strong, with \( \Lambda > 4 \). If we take the the typical magnetic field strength as it’s RMS value at the outer boundary of the shell, only three of the two hundred dynamo cases considered here have \( \Lambda > 4 \).

Another assumption employed here is that the Rayleigh number at which convection onsets follows a theoretical scaling for rapidly rotating, non-magnetic convection in a sphere, \( Ra_c \propto E^{-4/3} \) (Roberts (1968), Busse (1970), Jones et al. (2000), Dormy et al. (2004)). In the presence of strong magnetic fields, the critical Rayleigh number may scale differently, as \( Ra_c \propto E^{-1} \) (Zhang (1995)). Here, we assume that the critical Rayleigh number applicable to the empirical, rapidly rotating heat transfer scaling law, \( Nu = (Ra/Ra_c)^{6/5} \), is the non-magnetic critical Rayleigh number, \( Ra_c \propto E^{-4/3} \).

We hypothesize that the ideas of boundary layer control of heat transfer regimes
demonstrated in the non-magnetic, plane layer convection study will apply to planetary dynamo models. These ideas include: the existence of two separate regimes, rapidly rotating and weakly rotating, with distinct empirical heat transfer scalings in each regime, $Nu \sim (Ra/Ra_c)^{6/5}$ and $Nu \sim Ra^{2/7}$, respectively; regime transitions determined by relative boundary layer thicknesses and described by the transitional Rayleigh number, $Ra_t = E^{-7/4}$. (Note that the lack of a priori knowledge of the prefactor for a non-rotating $Nu \propto Ra^{2/7}$ scaling law renders the prefactor for $Ra_t$ found in the plane-layer study meaningless. Hence, we take $Ra_t = E^{-7/4}$ for the remainder of this document).

Here, we test this hypothesis using a broad array of numerical dynamo models.

### 4.1.2 Numerical Model Setup

I analyze a dataset consisting of two hundred dynamo simulations. The simulations are produced by Johannes Wicht’s MagIC code (Wicht (2002)), which models three-dimensional convection of a Boussinesq, electrically-conducting fluid in a spherical shell rotating about the axial $\hat{z}$-direction with fixed angular velocity $\Omega$. The shell has isothermal boundaries with an imposed temperature contrast of $\Delta T$ between inner and outer boundaries. Gravity is assumed to vary linearly with radius and has a value of $g_o$ on the outer boundary. Figure 52 shows a schematic depiction of the dynamo model’s geometry. See appendix A.3 for further details on numerical dynamo methods.
Figure 52: A schematic illustration of the dynamo model geometry. The convecting fluid is contained in a spherical shell between a hot inner boundary and cold outer boundary, $T_i > T_o$. Gravity points radially inward and varies linearly with radius. The rotation axis is vertical with uniform non-dimensional rotation rate $E^{-1}$.

This data set consists of two subsets. The first consists of twenty-one simulations (hereon called the dynamo subset models) computed in house by Krista Soderlund. The second data set comes from collaborator Ulrich Christensen, made up of one hundred seventy nine additional dynamo cases (hereon called the dynamo factory models), most of which were previously reported (e.g., Christensen and Aubert (2006), Olson and Christensen (2006)). We have greater diagnostic capabilities for the in-
Table 3: Typical nondimensional parameters used in dynamo modeling, with estimates for Earth's core for comparison. Here, $\alpha_T$ is the coefficient of thermal expansion, $g$ is gravitational acceleration, $\Delta T$ is the temperature drop from inner to outer boundary (thus, $\alpha_T g \Delta T$ represents the anomalous gravitational acceleration of a fluid parcel with temperature anomaly $\Delta T$), $L$ is the shell thickness, $\nu$ is the viscous diffusivity, $\kappa$ is the thermal diffusivity, $\Omega$ is the background rotation rate, $\eta$ is magnetic diffusivity, $q_{sa}$ is the superadiabatic heat flow per unit area, $k$ is the fluid’s thermal conductivity, and $U$ is a typical fluid velocity.

<table>
<thead>
<tr>
<th>Term</th>
<th>Explanation</th>
<th>Definition</th>
<th>Earth’s Core</th>
<th>Dynamo Factory Models</th>
<th>Dynamo Subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ra$</td>
<td>Buoyancy Diffusion</td>
<td>$\frac{\alpha_T g \Delta T L^3}{\nu \kappa}$</td>
<td>$10^{20} - 10^{30}$</td>
<td>$3 \times 10^5 \leq Ra \leq 2.2 \times 10^9$</td>
<td>$2 \times 10^6 &lt; Ra &lt; 5 \times 10^8$</td>
</tr>
<tr>
<td>$E$</td>
<td>Coriolis Viscosity</td>
<td>$\frac{\nu}{2 \Omega L^2}$</td>
<td>$\sim 10^{-15}$</td>
<td>$5 \times 10^{-7} \leq E \leq 5 \times 10^{-4}$</td>
<td>$5 \times 10^{-6} \leq E \leq 10^{-4}$</td>
</tr>
<tr>
<td>$Pr$</td>
<td>Viscous Diffusion</td>
<td>$\frac{\nu}{\kappa}$</td>
<td>$10^{-1}$</td>
<td>$0.1 \leq Pr \leq 30$</td>
<td>$1 \leq Pr \leq 2$</td>
</tr>
<tr>
<td>$Pm$</td>
<td>Viscous Diffusion</td>
<td>$\frac{\nu}{\eta}$</td>
<td>$\sim 10^{-6}$</td>
<td>$0.06 \leq Pm \leq 20$</td>
<td>$1 \leq Pm \leq 2.5$</td>
</tr>
<tr>
<td>$Nu$</td>
<td>Total Heat Flux</td>
<td>$\frac{q_{sa} L}{k \Delta T}$</td>
<td>$1 &lt; Nu &lt; 32$</td>
<td>$2.5 &lt; Nu &lt; 25$</td>
<td></td>
</tr>
<tr>
<td>$Ra_t$</td>
<td>Boundary Layer Crossing</td>
<td>$E^{-7/4}$</td>
<td>$0.013 &lt; Ra/Ra_t &lt; 40$</td>
<td>$0.042 \leq Ra/Ra_t &lt; 42$</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td></td>
<td>$Nu \propto Ra^\alpha$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Re$</td>
<td>Inertia Viscosity</td>
<td>$\frac{UL}{v}$</td>
<td>$1 \leq Re &lt; 3 \times 10^3$</td>
<td>$10 \leq Re &lt; 3 \times 10^3$</td>
<td></td>
</tr>
</tbody>
</table>

house, dynamo subset models. The ranges of parameters explored by each of these datasets are given in table 3. All simulations have an Earth core-like geometry ($\chi \equiv R_i/R_o = 0.4$ for dynamo subset models, and $\chi = 0.35$ for dynamo factory models, where $R_i$ and $R_o$ are inner and outer boundary radii, respectively) and rigid, no-slip boundary conditions. While the dynamo subset models are limited to isothermal boundaries, the dynamo factory models have both isothermal and constant heat flux thermal boundary conditions (Sakuraba and Roberts (2009)).
Figure 53: Heat transfer behavior, \( Nu \) versus \( Ra \), from a broad array of convective dynamo models (see table 1 for parameter ranges). The Nusselt number, \( Nu \). The non-rotating scaling law, \( Nu_{\text{Non-Rotating}} = 0.075Ra^{2/7} \), is shown as a solid line. The rapidly rotating scaling law, \( Nu \propto Ra^{6/5} \), is shown as a dashed line. The majority of the dynamo model data shown are provided by U. R. Christensen, and are supplemented by additional cases carried out by K. Soderlund.

4.1.3 Heat Transfer Results

Figure 53 shows heat transfer data, \( Nu \) versus \( Ra \), for all dynamo cases. The basic behavior of the heat transfer data are similar to that observed in plane layer studies (e.g., Rossby (1969), King et al. (2009), Schmitz and Tilgner (2009)). Convective heat transfer is strongly affected by the background rotation, whose strength is characterized by \( E^{-1} \). The stabilizing influence of the Coriolis force requires models with higher
rotation rates to be driven harder in order to convect. Thus, the onset of convection is
delayed by the influence of rotation as $E$ decreases. When convection begins, $Nu$ in-
creases more rapidly than would be expected for non-rotation convection. Here, heat
transfer data are well described by a $Nu \propto Ra^{6/5}$ scaling (dashed line). In the higher
$E$ cases, heat transfer behavior transitions to a more shallow scaling for sufficiently
high $Ra$. Here, heat transfer data appear to asymptote to a $Nu \approx 0.075Ra^{2/7}$ scaling
(solid line). Thus, within particular parameter ranges, the data are well described
by either the rapidly rotating or non-rotating scaling exponents found in the plane
layer study. As mentioned previously, the goal of heat transfer scalings is to collapse
the data in figure 53 through some combination of control parameters, such that a
power-law scaling may be extrapolated to the extreme planetary settings. In order
to collapse the observed rapidly rotating heat transfer behavior to a single scaling,
consideration of importance of rotation is necessary.

Figure 54 shows $Nu$ normalized by the non-rotating scaling ($Ra^{2/7}$) versus $Ra$
normalized by its predicted transitional value, $Ra_t = E^{-7/4}$, for all dynamo cases. The
boundary layer transition scaling effectively collapses the broad array of dynamo heat
transfer data. The scaling regimes hinted at by figure 53 are more clearly observed in
this parameter space defined by the boundary layer controlled transitional Rayleigh
number. Well into the rapidly rotating regime ($Ra \lesssim 0.1Ra_t$), heat transfer behavior
follows the $Nu = (Ra/Ra_c)^{6/5}$ scaling (dashed line). In the weakly rotating regime
($Ra > Ra_t$), heat transfer transitions to a $Nu \sim Ra^{2/7}$ scaling (solid horizontal line).
Figure 54: Heat transfer transitions from the broad array of convective dynamos. Symbols are the same as in figure 53. The Nusselt number, \( \text{Nu} \), is normalized by the non-rotating scaling \( Ra^{2/7} \), and is shown versus the Rayleigh number normalized by the transitional Rayleigh number, \( Ra/Ra_t = Ra E^{7/4} \). The solid horizontal line represents a \( Nu = 0.075 Ra^{2/7} \) non-rotating scaling law. The dashed line represents the rapidly rotating scaling law \( Nu = (Ra/Ra_c)^{6/5} \), where \( Ra_c \propto E^{-4/3} \) (Jones et al. (2000)). The majority of the dynamo model data shown are provided by U. R. Christensen, and are supplemented by additional cases carried out by K. Soderlund.

Interestingly, the data again show no strong dependence on the Prandtl numbers, \( Pr \) and \( Pm \). Thus, despite the added complexities of the spherical geometry and
Figure 55: Boundary layer thicknesses versus the transition parameter $Ra/Ra_t$ from the dynamo subset ($10^{-5} \leq E \leq 10^{-4}$, $2 \times 10^7 < Ra < 5 \times 10^8$, $1 \leq Pr \leq 2$, and $1 \leq Pm \leq 2.5$). The velocity boundary layer (an Ekman-Hartmann layer) thickness is shown with upward pointing triangles, and is defined as the radial location of the first local maximum in rms velocity above the model’s inner boundary (Belmonte et al. (1994), King et al. (2009)). The thermal boundary layer is shown with downward pointing triangles and is defined as the radial location of the first local maximum in temperature variance above the model’s inner boundary (Belmonte et al. (1994), King et al. (2009)). The distance between errorbars represents the local spatial resolution. The two boundary layers cross near $Ra = Ra_t$, as predicted. Boundary layer thickness measurements are provided by K. Soderlund.

magnetohydrodynamic effects, the heat transfer regimes identified in plane layer, non-magnetic convection are also evident in the planetary dynamo models.
As in the plane layer study, we again compare this transition in heat transfer behavior to measurements of the boundary layers themselves. Figure 55 shows the velocity and thermal boundary layer thicknesses from the dynamo subset plotted versus the transition parameter, $Ra/Ra_t$. The velocity boundary layer thickness is defined as the radial location of the first local maximum in rms velocity above the model’s inner boundary (Belmonte et al. (1994), King et al. (2009)). Similarly, the thermal boundary layer is defined as the radial location of the first local maximum in temperature variance above the model’s inner boundary (Belmonte et al. (1994), King et al. (2009)). The added complication of magnetic fields discussed above does not influence our measurement of viscous boundary layer thickness as this measurement is made directly from the velocity field (King et al. (2009)). Figure 55 shows that the two boundary layers cross where $Ra \approx Ra_t$. Thus, as predicted by the plane layer, non-magnetic convection study, the transition from rapidly rotating to non-rotating heat transfer behavior in the present dynamo models also corresponds to the interchanging of these two nested boundary layers.

### 4.1.4 Heat Transfer Scaling Regimes

Boundary layer control of heat transfer behavior in planetary dynamo models allows us to predict the limits of applicability of the various heat transfer scaling laws. In the weakly rotating regime, $Ra < Ra_t$, the data conform to a $Nu = 0.075Ra^{2/7}$ scaling. This hints at the possibility of a non-rotating, turbulent convective heat transfer
scaling law of $\alpha \approx 2/7$ in the spherical geometry, like that found in plane-layer convection. Non-rotating, plane-layer convection has the advantage of being carried out both numerically and in the laboratory. Laboratory convection studies are historically capable of achieving higher degrees of supercriticality than numerical models (e.g., Niemela et al. (2000)). Because of the difficulty in producing a spherically radial gravity field in the laboratory, high Rayleigh number, non-rotating convection has not been well-explored in this geometry (e.g., Tilgner (1996)). The scaling prefactor, 0.075, although not well-constrained, is lower than that found in the plane layer studies, which is likely due to geometric differences. That the heat transfer transition from $\alpha \approx 6/5$ to $\alpha \approx 2/7$ seen in plane layer studies is also observed in the present spherical dynamo models allows us to hypothesize that the $Nu \propto Ra^{2/7}$ scaling law applies to the dynamo regions of slowly rotating planets and stars ($Ra > Ra_t$).

In the rapidly rotating regime, $Ra < Ra_t$, heat transfer data are well described by the $Nu = Ra^{6/5}$ scaling law found in the plane layer study. Assuming $Ra_c = C_1 E^{-4/3}$, this scaling can be rewritten in terms of the axis parameters in figure 54 as

$$NuRa^{-2/7} = (Ra/Ra_c)^{6/5}Ra^{-2/7} = C_1^{-6/5}Ra^{32/35}E^{8/5} = C_2 (Ra/Ra_t)^{32/35}$$

This scaling is represented by the dashed line in figure 54. The scaling prefactor $C_2$ is determined by the fit of the $NuRa^{-2/7} = C_2 (Ra/Ra_t)^{32/35}$ scaling to the dynamo data, yielding $C_2 = 0.225$. This corresponds to $Nu = (Ra/Ra_c)^{6/5}$ with $C_1 = C_2^{-5/6} = 3.5$. This determination of $C_1$ is empirical and based on the assumptions that $Nu =$
\((Ra/Ra_c)^{6/5}\) in the rapidly rotating regime with \(Ra_c = C_1 E^{-4/3}\). We can compare the prefactor \(C_1\) to theoretical values from Jones et al. (2000), who find \(1 \lesssim C_1 \lesssim 8\) for \(0.1 \leq Pr \leq 10\). Thus, although here we ignore the Prandtl number dependence of the critical Rayleigh number, the empirically determined prefactor \(C_1 = 3.5\) is in first-order agreement with that from the theoretical analysis of rotating convection.

It is important to mention here that we currently have access to less than two orders of magnitude in \(Ra\) for a given \(E\) within the rapidly rotating regime. It is therefore difficult to be certain these results are asymptotic for \(E \downarrow 0\). The range of \(Ra\) accessible within the rapidly rotating regime can be written as

\[
Ra_t/Ra_c \sim E^{-5/12}.
\]  

(92)

In order to test the validity of scaling laws derived from numerical models, lower Ekman number simulations must be employed so that a greater range of supercritical, sub-transtional convection is accessed.

As mentioned in section 3.1, another heat transfer scaling law has been developed for rotating convection and dynamo models such that, following Kolmogorov’s second hypothesis of similarity (Kolmogorov (1941)), it is independent of molecular diffusivities (Christensen (2002), Christensen and Aubert (2006)). The authors use the rotation period as the dominant timescale to develop diffusivity-free parameters

\[
Ra^*_Q = RaN u E^3 Pr^{-2}
\]

(93)

\[
Nu^* = Nu E Pr^{-1},
\]

(94)
and seek a scaling

\[ Nu^* \propto Ra_Q^{*\beta}. \]  \hspace{1cm} (95)

Empirically, they find \( \beta \approx 6/11 \) (Christensen (2002), Christensen and Aubert (2006)). This diffusivity-free parameter space, however, tends artificially to spread the heat transfer data rather than collapse it, which may introduce ambiguity in its applicability (Aurnou (2007)). Furthermore, the range of parameter space in which the rotational timescale should be dominant is not quantified, and so it is unclear for what range of parameters diffusivity-free scaling should apply.

We compare the diffusivity free scaling law with the heat transfer results discussed above. For comparison, the diffusivity free scaling law and the rapidly rotating plane layer scaling law can be rewritten in terms of base parameters as

\[ Nu^* \propto Ra_Q^{*6/11} \rightarrow Nu \propto Ra^{1.2}E^{1.4}Pr^{-0.2}, \]  \hspace{1cm} (96)

\[ Nu = (Ra/Ra_c)^{6/5} \rightarrow Nu \propto Ra^{1.2}E^{1.6}, \]  \hspace{1cm} (97)

respectively. Thus, despite minor differences, these two scalings are not easily distinguishable, especially within the currently limited range of accessible parameter space. Since the boundary layer control hypothesis predicts the bounds for rotationally controlled behavior, however, we surmise that the diffusivity free scaling law, which takes the rotation period as its dominant timescale, should also hold only for \( Ra < Ra_c \).

Figure 56 shows heat transfer data normalized by the diffusivity free scaling (equation (96)) in terms of the boundary layer predicted transition. Rather than spreading the data across many orders of magnitude, as in \( Nu^*-Ra_Q^* \) plots, here we collapse
Figure 56: Nusselt number normalized by the diffusivity free scaling from equation (96) versus the boundary layer transition parameter, $Ra/Ra_t$. The horizontal line represents the diffusivity free scaling, $Nu^* = 0.1Ra_Q^{6/11}$. As for the rapidly rotating scaling law from figure 54, the range of applicability of the diffusivity free scaling is limited to the rapidly rotating regime defined by $Ra < Ra_t$. The majority of the dynamo model data shown are provided by U. R. Christensen, and are supplemented by additional cases carried out by K. Soderlund.

the data to illustrate the limits of the scaling law’s applicability. The data are well described by a $Nu^* = 0.1Ra_Q^{6/11}$ scaling within the boundary layer controlled rapidly rotating convection regime ($Ra < Ra_t$). As the models move into the weakly rotating regime ($Ra > Ra_t$), the heat transfer data diverge from the diffusivity free scaling. Thus, we propose the transitional Rayleigh number as an upper bound for
the applicability of this scaling.

The diffusivity-free heat transfer scaling (equation (96)) is but one member of a suite of empirical, diffusivity-free scaling laws. Within the range of parameter space currently accessible, this rotationally dependent diffusivity-free approach has produced successful scalings for several other important model outputs such as typical velocities, length scales, and magnetic field strengths (Christensen (2002), Aubert (2005), Olson and Christensen (2006), Christensen et al. (2009)). These scaling laws implicitly take the rotation period as their dominant timescale, replacing the typical diffusion timescales present in the base parameters (table 3). We observe heat transfer behavior that is no longer controlled by rotation rate when $Ra > Ra_t$ (figures 54, 56). As shown for the diffusivity free heat transfer scaling, we further postulate that $Ra_t$ is an upper bound for the application of the other diffusivity-free scaling laws of Aubert (2005), Christensen and Aubert (2006), Olson and Christensen (2006), Christensen et al. (2009).

4.2 Implications of Boundary Layer Transitions in Dynamo Models

4.2.1 Mixing and Mean Thermal Profiles

The rapidly rotating and weakly rotating heat transfer regimes exhibit different global thermal signatures. Figures 57 and 58 illustrate the transition in the mean temperature structures between the regimes. The rapidly rotating regime maintains large-
Figure 57: Meridional slices of time-averaged temperatures \(((T+T_o)/\Delta T)\) from the dynamo subset. Rapidly rotating cases \((Ra < Ra_t; a, b)\) are constrained by the Coriolis force, permitting large-scale temperature gradients. As the influence of rotation is lost \((Ra > Ra_t; c, d)\), the bulk fluid becomes well-mixed and strong mean thermal gradients survive only in the boundary layers. Dynamo subset cases are carried out by K. Soderlund.

scale mean internal temperature gradients while the weakly rotating regime undergoes vigorous mixing that produces a statistically isothermal interior fluid. (Mixing is a scale-dependent concept; here we consider mixing on the largest length scales of the system.) This isothermalization has been observed in non-rotating (Belmonte et al. (1994), Camussi and Verzicco (1998)) and weakly rotating convection (Aurnou et al. (2007), King et al. (2009)), and is typically associated with chaotic, three-dimensional turbulence. In rapidly rotating convection, however, the persistence of coherent, large-scale structures prevents strong mixing, thereby permitting the establishment of significant non-zero mean temperature gradients within the bulk fluid,
Figure 58: Mean radial temperature profiles from the dynamo subset cases shown in figure 57. Temperature is normalized as \((T - T_o)/\Delta T\), and radial position is normalized by the outer boundary radius. The dashed line indicates the conductive temperature profile (equation (98)). Internal temperature gradients approach zero as \(Ra/Ra_t\) increases, which indicates the increasing efficiency of thermal mixing as we move from the rapidly rotating regime to the weakly rotating regime. Dynamo subset cases are carried out by K. Soderlund.

even for high \(Ra\) (Gillet and Jones (2006), Sprague et al. (2006), King et al. (2009)).

In order to quantify the thermal mixing more systematically, we measure the mean temperature lapse rate at mid-shell for each of the dynamo subset models. A strongly
Figure 59: Temperature lapse rate, $|\frac{dT}{dr} \frac{R_o - R_i}{\Delta T}|$, at mid-shell ($r = (R_i + R_o)/2$) versus the boundary layer transition parameter, $Ra/Ra_t$, for the dynamo subset. The dashed line represents the conductive lapse rate at mid-shell (from equation (98)): $\frac{dT}{dr} = -\Delta T \frac{R_o R_i}{R_o - R_i} r^{-2} = -0.82 \frac{\Delta T}{R_o - R_i}$. We observe that the lapse rate decreases as $Ra/Ra_t$ increases. As the thermal boundary layer becomes thinner than the velocity boundary layer, strong mixing in the bulk fluid reduces the internal temperature gradient to nearly zero (isothermal). Dynamo lapse rate data are provided by K. Soderlund.

well-mixed Boussinesq fluid will have a zero mean temperature gradient at mid-shell, $dT/dr |_{\text{mid-shell}} = 0$. As the influence of rotation grows (for decreasing $Ra/Ra_t$), some rigidity is imparted on the flow via the Coriolis force, and increasingly strong ther-
mal gradients can abide, as seen in the plane layer study. Through measurements of mid-shell lapse rates in the dynamo subset shown in figure 59, we observe a transformation from a nearly conductive thermal signature in the rotationally controlled regime \((Ra < Ra_t)\) to a nearly isothermal interior fluid in the weakly rotating regime \((Ra > Ra_t)\). This illustrates that the transition to non-rotating heat transfer behavior \((Nu \sim Ra^{2/7})\) is related to the elimination of mean interior temperature gradients.

We also observe evidence for boundary layer controlled isothermalization in the broader array of dynamo factory models. Since we don’t have access to internal temperature field data in the dynamo factory models, we make a proximate measure of isothermalization by way of mean temperature data. We can relate the mean temperature measured over the volume of the shell to the efficiency of mixing by comparing these data to the two end-member cases: conductive and isothermal.

First, in an incompressible spherical shell with isothermal boundaries, the conductive case will have a temperature profile of the form:

\[
T(r) = T_o + \frac{R_o R_i}{R_o - R_i} \left( \frac{1}{r} - \frac{1}{R_o} \right) \Delta T,
\]

where \(R_o\) and \(R_i\) are the radii of the outer and inner boundaries, respectively, and \(T_o\) and \(T_o + \Delta T\) are the imposed outer and inner boundary temperatures, respectively (Kono and Roberts (2001)). We can integrate equation (98) over the volume of the shell to solve for the mean temperature:

\[
\langle T_{\text{conductive}} \rangle = T_o + \frac{\Delta T}{2} \left( \frac{R_o + 2R_i}{R_o^2 + R_o R_i + R_i^2} \right).
\]
This expression can also be written in terms of the shell’s radius ratio, \( \chi = R_o/R_i \) as

\[
\langle T_{\text{conductive}} \rangle = T_o + \frac{\Delta T}{2} \left( \frac{\chi(1 + 2\chi)}{1 + \chi + \chi^2} \right).
\] (100)

With the dynamo factory model’s radius ratio of \( \chi = 0.35 \), we then predict an average temperature of

\[
\langle T_{\text{conductive}} \rangle = T_o + 0.21\Delta T
\] (101)

for the conductive case.

Second, we can estimate the mean temperature from a well-mixed temperature profile. In this configuration, we assume the fluid is composed of two regions: the boundary layers (inner and outer), and the interior fluid, also called the bulk fluid. The interior fluid is isothermal and makes up the vast majority of the fluid by volume. We also assume that inner and outer thermal boundary layers have equal thickness, which we qualitatively observe to hold true in dynamo subset velocity and temperature profiles. The temperature drop across the inner and outer thermal boundary layers is \( \Delta T_i \) and \( \Delta T_o \), respectively. Conservation of outward heat flux through concentric spherical surfaces dictates that

\[
\frac{\Delta T_i}{\Delta T_o} = \left( \frac{R_o}{R_i} \right)^2.
\] (102)

An isothermal interior further implies the entirety of the temperature drop occurs in the thin boundary layers:

\[
\Delta T_i + \Delta T_o = \Delta T;
\] (103)
and the interior temperature will be that at the bulk edge of the thermal boundary layers:

\[ T_{\text{interior}} = T_o + \Delta T_o = (T_o + \Delta T) - \Delta T_i. \]  

(104)

Manipulating equations (102) - (104) then gives temperature of the interior fluid to be:

\[ T_{\text{interior}} = T_o + \left( \chi^2 + 1 \right)^{-1} \Delta T. \]  

(105)

Thus, shell aspect ratio of \( \chi = R_i/R_o = 0.35 \) gives an interior fluid temperature of

\[ T_{\text{interior}} = T_o + 0.11 \Delta T. \]  

(106)

If we ignore the contribution to mean volume temperature by that within the small boundary layers, the mean shell temperature for the isothermal case is the temperature to which the interior isothermalizes:

\[ \langle T_{\text{isothermal}} \rangle = T_{\text{interior}}. \]  

(107)

Figure 60 illustrates how each of these two end member predictions varies with the radius ratio. Figure 61 shows average temperature measurements from the dynamo factory models in comparison with the end-member predictions. The mean internal temperature is the average temperature over the volume of the shell, \( \langle T \rangle - T_o/\Delta T \), and serves as a proximate measure of internal temperature gradients (figure 59). We observe a transition from nearly conductive to well mixed mean temperatures as we move from the rapidly rotating regime to the weakly rotating regime. For \( Ra < Ra_t \),

150
Figure 60: Theoretical mean shell temperature versus the radius ratio, $\chi = R_i/R_o$. The mean temperature is the average temperature over the volume of the shell, $(\langle T \rangle - T_0)/\Delta T$. The solid (dashed) line represent predicted mean temperature from equation (100) (equation (105)). End cases are: a full sphere, $\chi = 0$; and a plane layer, $\chi \approx 1$. The gray region indicates that occupied by the present geodynamo models.

Internal temperatures demonstrate nearly conductive profile behavior (dotted line), evidence for the persistence of strong, large scale thermal gradients in the bulk fluid. Warmer than conductive average temperatures are due to the warming of the outer region of the shell (where the bulk of the fluid volume resides) relative to a conductive profile near the onset of convection (see $Ra/R_{a_t} = 0.14$ case in figure 58). When $Ra > R_{a_t}$, the thermal signature is that of a well-mixed, nearly adiabatic profile (dashed line). This illustrates the increasing efficiency of thermal mixing as the influence of rotation is diminished in planetary dynamo models.

Convective regions in planets and stars are typically assumed to be well-mixed
Figure 61: Mean internal temperature versus the boundary layer transition parameter, $Ra/Ra_t$, from the dynamo factory dataset (provided by U. R. Christensen). Symbols are the same as in figure 53. The mean internal temperature is the average temperature over the volume of the shell, $(\langle T \rangle - T_o)/\Delta T$, and serves as a proximate measure of internal temperature gradients. Mean temperatures near the conductive estimate (dotted line, $T = 0.21$) are considered poorly mixed. (Warmer than conductive mean temperatures are likely due to warmer than conductive temperatures in the outer region of the shell, where the majority of the fluid volume resides (figure 58).) Those near the isothermal estimate (dashed line, $T = 0.11$) are considered well mixed. We observe that models with increasing $Ra/Ra_t$ have cooler average temperatures, indicative of the increasing proficiency of thermal mixing.
on account of their strongly turbulent nature. The Reynolds number, $Re$, compares inertial and viscous forces on global length scales, and is thought of as a parameterization of the degree of turbulence (see table 1 for definition). High $Re$ fluids in the absence of rotation are dispersive media wherein gradients of passive scalars are quickly smoothed (McWilliams (2006)). We may expect, then, that the thermal...
mixing observed in the dynamo models may be strongly influenced by \( Re \). Figure 62 shows the mean temperature measurements as in figure 61, but now plotted against the Reynolds number. We observe no dependence of mixing on \( Re \). It is the breakdown of rotational control (figure 61), not the Reynolds number, that permits the strong mixing of the convective fluid in dynamo models.

### 4.2.2 Magnetic Field Morphology Transitions

We further hypothesize that the two convective regimes will generate magnetic fields with different morphologies. We expect the subtransitional regime, with its penchant for large scale, symmetric convection features, will generate strongly dipolar fields (e.g., Kuang and Bloxham (1997)). Non-dipolar fields, however, are historically less common in models (e.g., Stanley and Bloxham (2004)). As the rotationally imposed organization is destroyed in the supertransitional regime, we anticipate that convection may generate disorganized, smaller-scale magnetic field structures (e.g., Olson and Christensen (2006)). Figure 63 shows examples of velocity and magnetic fields on each side of the heat transfer transition \( \left( \frac{Ra}{Ra_t} \approx \frac{Ra_t}{Ra_t} \right) \). The case on the left, for which \( \frac{Ra}{Ra_t} = 0.15 \), displays coherent, axially aligned flow structures and a strongly dipolar magnetic field. The case on the right, for which \( \frac{Ra}{Ra_t} = 15 \), shows the breakdown of this quasigeostrophic behavior, with strongly three-dimensional flow and a non-dipolar magnetic field.

To analyze the dynamo data set more systematically, we examine the models’
Figure 63: Examples of velocity and magnetic fields from the dynamo subset models. The top row shows meridional slices of snapshots of radial velocity, with the color scale set to illustrate the models’ peak values for comparison of flow structures (not magnitude). The bottom row shows snapshots of magnetic field strength at the models’ outer boundary, again with the color scale set to illustrate the models’ peak values. The case on the left has $Ra = 5 \times 10^6$ and $E = 5 \times 10^{-5}$, and therefore $Ra/Ra_t = 0.15$. The case on the right has $Ra = 5 \times 10^8$ and $E = 5 \times 10^{-5}$, and therefore $Ra/Ra_t = 15$. Dynamo subset models are carried out by K. Soderlund.

*dipolarity*. Dipolarity is defined as the mean magnitude of the dipolar component of
the magnetic field at the outer boundary normalized by that of all field components,

\[
\text{Dipolarity} = \frac{\langle |B_{l=1}(r = R_o)| \rangle}{\langle |B_{1 \leq l \leq l_{\text{max}}}(r = R_o)| \rangle},
\]

where \( l \) is the spherical harmonic degree in latitude, \( l_{\text{max}} \) is the maximum value of \( l \) resolved by the numerical model, and \( R_o \) is the outer boundary radius. If the boundary layer controlled heat transfer regime transitions are linked to magnetic field generation, we expect dipolarity to decrease with increasing \( Ra/Ra_t \).

Figure 64 shows dipolarity versus \( Ra/Ra_t \) for the entire dynamo dataset. Our transition scaling describes a transformation from strongly dipolar fields (\( Ra/Ra_t \ll 1 \)) to smaller scale, non-dipolar magnetic fields (\( Ra/Ra_t \gg 1 \)). Clearly, the transition is not perfectly sharp, and a detailed investigation of dynamics near the transition is needed. Notably, previous attempts to describe the transition from dipolar to multi-polar dynamos have been complex permutations of input and output parameters (e.g., Olson and Christensen (2006)). Here, instead, we attempt to describe this transition in magnetic field morphology using only the Rayleigh and Ekman numbers, both of which are input parameters. Furthermore, where previous scaling descriptions of this transition have been empirical, ours is based on an underlying physical mechanism adapted from hydrodynamic, plane layer rotating convection.

Dynamo theory typically separates magnetic field generation mechanisms into one of two categories: mean field dynamos, and fluctuating dynamos. Mean field dynamos generate magnetic fields at length scales larger than the typical kinetic length scales (Schekochihin et al. (2005)). This separation of scales is usually attributed to helical
Figure 64: Magnetic field dipolarity versus $Ra/Ra_t$. Symbols are the same as in figure 53 (with color denoting Ekman number). Dipolarity is defined as the mean magnitude of the dipolar component of the magnetic field at the outer boundary normalized by that of all field components (see equation (108)). The majority of the dynamo model data shown are provided by U. R. Christensen, and are supplemented by additional cases carried out by K. Soderlund.

flow, which leads to non-zero correlations of small scale turbulence on mean field averages (Käpylä et al. (2009)). Helical flow is a common feature of rotating convection, as seen in section 3.4. Fluctuating dynamos, also called small-scale dynamos, turbulent dynamos, or fast dynamos, amplify magnetic energy randomly at length scales below the energy containing scale of the turbulence (Schekochihin et al. (2005)). Figure 65
Figure 65: Length scales of magnetic field generation versus the transition parameter. The color scale denotes dipolarity (equation (108)), not the Ekman number as in previous figures. The length scales (magnetic and kinetic) are defined as the inverse of the spherical harmonic degree, \( l \), at which the energies (magnetic and kinetic) are observed to peak in power spectra. The transition from dipolar to non-dipolar magnetic fields is observed to be connected to a transition from mean field to fluctuating dynamos. The majority of the dynamo model data shown are provided by U. R. Christensen, and are supplemented by additional cases carried out by K. Soderlund.

shows the ratio of magnetic length scales to kinetic length scales from the dynamo models. These length scales are defined as the inverse of the latitudinal spherical harmonic degree at which energy, kinetic or magnetic, is highest. We observe that
the transition from dipolar to non-dipolar magnetic fields near the boundary layer transition occurs roughly where the ratio of these two length scales crosses unity. That is, the magnetic field morphology transition seen in figure 64 is linked to the transition between these two types of dynamos. When $Ra \lesssim Ra_t$, the magnetic scales are typically larger than the kinetic scales, and so the dipolar dynamos are typically mean field dynamos. Conversely, when $Ra \gtrsim Ra_t$, the magnetic scales are typically smaller than the kinetic scales, and so the non-dipolar dynamos are typically fluctuating dynamos. This indicates a transition in field generation mechanisms between the two regimes.
5 Core Applications of Boundary Layer Control
5.1 Core Convection Regime Estimate

In this chapter, I apply the idea of boundary layer controlled convection regimes to Earth’s core. In order to apply these results to the core, we must define the transition scaling in terms of geophysical observables. The Rayleigh number, $Ra$, is dependent on a system’s global density gradient (in thermal convection, the temperature gradient), which is often difficult to observe in nature. The flux-Rayleigh number, $Ra_f = Ra Nu$, depends instead on the overall buoyancy flux. For thermal convection, $Ra_f = (\alpha T g L^4 q)/(\rho C_p \kappa^2 \nu)$, where $q$ is the heat flux. Combining $Ra_t = E^{-7/4}$ and $Nu_t = E^{-1/2}$, we can acquire a transitional flux-Rayleigh number, $Ra_{ft} = Ra_t Nu_t = E^{-9/4}$. Thus we can determine the convective regime of a given system provided we know the fluid properties, system size and rotation rate, as well as the emitted heat flux.

A typical estimate of the Ekman number in Earth’s liquid metal outer core is $E \approx 10^{-15}$ (e.g., Sumita and Olson (2000), Christensen and Aubert (2006)), which allows us to estimate a transitional flux-Rayleigh value of $Ra_{ft} \approx 6 \times 10^{33}$ for the core, below which we expect rotation to control convective heat transfer. Using the following estimates (de Wijs et al. (1998), Gubbins (2001)): $\alpha_T \approx 10^{-4}$ K$^{-1}$; $g \approx 10$ m/s$^2$, $L \approx 2 \times 10^6$ m; $\rho \approx 10^4$ kg/m$^3$; $C_p \approx 1000$ J/kgK, $\kappa \approx 10^{-5}$ m$^2$/s; $\nu \approx 10^{-6}$ m$^2$/s; and 4 TW total superadiabatic heat from the core (Nimmo et al. (2004)), corresponding to a superadiabatic $q \approx 2.6 \times 10^{-2}$ W/m$^2$; we estimate a flux Rayleigh number of $Ra_f \approx 6 \times 10^{29}$ in the core. Thus, we estimate that Earth’s core is in the
rapidly rotating regime, since $Ra_f < Ra_f^t$.

Using this estimate of $Ra_f = RaNu$ in the core, as well as the empirical, rapidly rotating heat transfer scaling law, $Nu \approx (Ra/Ra_c)^{6/5}$, we can solve for a novel estimate of the Rayleigh number in Earth’s core,

$$Ra_{\text{core}} \approx 7 \times 10^{24},$$

which corresponds to a superadiabatic Nusselt number, $Nu = Ra_f/Ra \approx 9 \times 10^4$. With $E \approx 10^{-15}$, the core has a transitional Rayleigh number of $Ra_{t\text{Core}} \approx 2 \times 10^{26}$. Thus, we can estimate that core convection occurs just below the boundary layer transition with

$$Ra_{\text{core}}/Ra_{t\text{Core}} \approx 10^{-1}. $$

This estimate of the convective state of Earth’s core places it very close to the boundary between the rapidly rotating and weakly rotating regimes. Because the Rossby number in the core is estimated to be very small ($Ro \approx 10^{-6}$, section 1.1), it is usually assumed that the core is nearly geostrophic (e.g., Bloxham and Jackson (1991)). However, we estimate (roughly) that $Ra_{\text{core}}/Ra_{t\text{Core}}$ is close to unity, which would place the core at the very limit of the applicability of rapidly rotating scalings observed in the models. This rough estimate of the convective regime for Earth’s core based on the boundary layer controlled transition parameters indicates that the influence of rotation on core convection may not be as dominant as typically thought.
5.2 Towards An Earth-like Geodynamo Model

As discussed in the introduction, two major observations of Earth’s magnetic field are that it is a nearly axially aligned dipole and that it reverses polarity sporadically. A major goal of geodynamo modeling is to obtain a simulation that is both dipolar and reversing. To date, the only models that have accomplished this are only dipolar on account of their strong, laterally heterogenous thermal boundary conditions (Kutzner and Christensen (2004)). I propose that the boundary layer transition parameters provide direction in seeking this Earth-like dynamo model.

Based on our rough estimate that $Ra_{\text{core}}/Ra_{t\text{core}} \approx 0.1$, we will assume here that the core value of $Ra/Ra_t$ is relatively close to unity, placing the core near the transition between the two regimes. Figure 64 shows that models operating near this location are typically dipolar. Earth’s magnetic field is estimated to have dipolarity $\lesssim 2/3$ (Bloxham and Jackson (1991), Christensen and Aubert (2006)). Although Earth’s field is dipolar (dipolarity $>0.5$), it is less dipolar than most of the numerical models for $Ra/Ra_t \approx 0.1$. This may indicate that rotational control in the core is not absolute. Furthermore, Earth’s field has been observed (indirectly) to reverse polarity with a typical time scale of roughly once every ten diffusion times (Glatzmaier et al. (1999)). Polarity reversals in dynamo models have been linked to the breakdown of rotational influence (Kutzner and Christensen (2002)). I speculate that core convection, whose magnetic field displays a rapidly rotating characteristic in it’s dipolarity, and a weakly rotating characteristic through it’s polarity reversals, occurs near the
Figure 66: Dynamo subset model reversal frequency versus the transition parameter, $Ra/Ra_t$.

The dipolarity of the models is illustrated by color. The emergence of reversing dynamos coincides with the transition from dipolar to multipolar dynamos. Dynamo subset models are carried out by K. Soderlund.

transition between rapidly rotating and weakly rotating convection.

Figure 66 shows observations of polarity reversals in the dynamo subset models. We find that the emergence of reversals is coincident with the transition from dipolar fields to multipolar fields. This supports the idea that reversals are made possible by
the decreasing influence of rotation. Since most of these models are computationally expensive to run, they are usually run for less than a single magnetic diffusion time. In order to find an Earth-like reversing, dipolar dynamo, we will need to choose a position in parameter space and run the simulation for many diffusion times. In order to do this, we must have a good idea where, in parameter space, to investigate. I propose that boundary layer controlled transition parameters indicate that Earth’s dynamo lies in the transitional region between truly dominant rotational control, where we observe strong dipoles but no reversals, and weakly rotation convection, where dynamos are no longer dipolar, but are able to reverse polarity.

5.3 Thermal State of the Core

Despite the uncertainty in determining the validity of specific scaling laws discussed in section 4.1.4, the steepness of the rapidly rotating heat transfer scaling (with $Ra$) appears to be robust for decreasing $E$. This quality provides basic insight into the relationship between the fluid dynamics and thermodynamics of dynamo regions such as Earth’s core. The convective flows in the core that generate the geomagnetic field are most likely driven by some form of buoyant driving, characterized by the Rayleigh number. The buoyant driving force can be related to the total heat escaping the core-mantle boundary as $Ra(Q_{sa})$, where $Q_{sa}$ is the superadiabatic heat power. We can isolate the thermodynamic source through the flux-Rayleigh number: $Ra_f = Ra Nu \propto Q_{sa}$, which is independent of the density gradient. If $Nu \propto Ra^\alpha$ for a given $E$, we
can write the relationship between fluid dynamic driving and thermodynamic source as $Ra \propto Q_{sa}^{1/(\alpha+1)}$. The steepness of the heat transfer behavior in the rapidly rotating regime ($\alpha > 1$) produces an effective buoyancy force that is only weakly dependent on the heat flux. For example, the current rapidly rotating scaling law, $\alpha = 6/5$, gives:

$$Ra \propto Q_{sa}^{5/11}.$$  \hspace{1cm} (111)

Convection systems residing in the non-rotating style regime, however, will exhibit a slightly stronger dependence of buoyancy force on heat flux. If we take $\alpha = 2/7$ in this regime, then

$$Ra \propto Q_{sa}^{7/9}.$$  \hspace{1cm} (112)

in slowly rotating bodies.

Figure 67 shows a schematic depiction of the relationship between buoyancy and heat flux in the two regimes. The weak dependence of buoyant driving on heat flux in the rapidly rotating regime is such that an order-of-magnitude change in global, superadiabatic heat flux will modify the buoyancy force by less than a factor of three. Assuming that Earth’s core resides in the rapidly rotating regime (equation (110)), we anticipate that convective driving is weakly dependent on heat flux in the core. This phenomenon may help explain the long-term stability of geomagnetic field strength despite the likelihood of strong global fluctuations in core-mantle boundary heat flux as the result of mantle convection (e.g., Tarduno et al. (2007), Courtillot and Olson (2007)).
Figure 67: A schematic of the relationship between effective buoyancy forcing ($Ra$) and superadiabatic heat flux ($Q_{sa}$) in logarithmic form. In the rapidly rotating regime, the fluid dynamically relevant forcing is weakly dependent on available convective heat flux.

The boundary layer controlled distinction between rotationally constrained convection ($Ra < Ra_t$) and well-mixed flows ($Ra > Ra_t$) discussed in section 4.2.1 may prove important for estimates of the gross thermal structures of the core and other convective regions of rotating planets and stars. The mean radial temperature profile of a convection system can be broken down into two components:

$$T(r) = T_a(r) + T_{sa}(r),$$  \hspace{1cm} (113)

where $T_a$ is the adiabatic temperature due to increasing pressure with depth, and $T_{sa}$ is the superadiabatic temperature. Boussinesq models, for example, have flat adiabats,
$dT_{a}/dr = 0$, so here an adiabatic profile is isothermal. Compressible bodies, such as Earth’s core, have significant adiabatic gradients. It is typically assumed that departures from adiabaticity are insignificant in the bulk fluid, $dT_{a}/dr \gg dT_{sa}/dr$, as the result of strong convective mixing (e.g., Stevenson (1987), Braginsky and Roberts (1995), Nimmo (2009)).

The rough estimate presented here gives $Ra_{core}/Ra_{t_{core}} \approx 10^{-1}$ for Earth’s core, where we have shown models produce more strongly superadiabatic temperature profiles. In this regime, much of the superadiabatic temperature drop is not in the thin boundary layers, as in the well-mixed regime, but is rather able to pervade the bulk fluid (e.g., figure 57 a, b). Since estimates of superadiabatic heat flux from the core are of the same order of magnitude as that conducted down the adiabat (Nimmo (2009)), our present results imply that potentially significant, large-scale superadiabatic density anomalies may survive in the core, which would not be permitted by considerations of turbulent mixing without rotation (Stevenson (1987)).

Interestingly, deviations from expected density profiles have been inferred seismically in the core (e.g., Soldati et al. (2003) Romanowicz and Breger (2000)), which disagree with the adiabatic PREM model (Dziewonski and Anderson (1981)). Notable features include thick (∼300 - 400 km) regions of anomalous seismic wave velocities at both the top (Souriau and Poupinet (1991)) and bottom (Kaneshima et al. (1994)) of the core. Furthermore, recent seismic studies have shown that observationally significant temporal fluctuations in density occur in Earth’s outer core (Dai and Song
The presence of these large scale regions of radial thermo-compositional stratification is at odds with the assumption that high $Re$ flows such as core convection must effectively mix the fluid (Gubbins et al. (2008)). In other words, according to many dynamical arguments, such density anomalies should not exist in the core (e.g., Stevenson (1987)).

It was shown in the previous chapter that large Reynolds numbers are not synonymous with strong mixing. I propose that, despite its high $Re$, flow in the core (for which $Ra \lesssim Ra_t$) is influenced by rotation enough so that mean superadiabatic density gradients may survive. In order to better understand the implications of this result as we extrapolate to planetary settings, systematic scalings of the lapse rate (figure 59) with control parameters are needed so that quantitative predictions can be made.
6 Conclusion
I have designed and built a unique experimental device, RoMag, that allows me to recreate conditions approaching those expected in Earth’s core and other extreme environments. With this device, I am able to study the complex, dynamic interaction of the three main planetary convection ingredients: buoyancy, rotation, and magnetic field.

The research presented in this document focuses on the fundamentals of how rotation influences convection dynamics. In order to investigate rotational control of convection systems, I’ve explored a wide range of parameters in search of where the influence of rotation breaks down. It is commonly assumed that the importance of rotational effects is determined by a balance between the global buoyancy and Coriolis forces. On the contrary, I have shown that the influence of rotation is controlled instead by very thin boundary layers near the edges of the fluid’s container. This represents a change in how we think of the realization of the Coriolis force in rotating fluids. Though the Coriolis force acts throughout the fluid, I find that its efficacy is decided at the fluid’s very edge.

I show this by demonstrating the existence of two separate regimes: a rapidly rotating regime; and a weakly rotating regime. In the rapidly rotating regime, heat transfer follows a $Nu = (Ra/Ra_c)^{6/5}$ scaling. In the weakly rotating regime, heat transfer abides by a non-rotating, $Nu \propto Ra^{2/7}$ scaling law. The transition is shown to occur where the Ekman and thermal boundary layers cross. When the Ekman layer is the thinner of the two, we observe rapidly rotating behavior. When the thermal
boundary layer is thinner, convection is weakly rotating. I use theoretical thickness scalings for these two boundaries to develop a scaling of the transition location, which occurs at $Ra = Ra_t \approx E^{-7/4}$.

The heat transfer transition is further shown to be linked to the breakdown of quasigeostrophic flow structures, or Taylor columns. As convection transitions from rapidly rotating to weakly rotating, these large scale features are replaced by more strongly three-dimensional, turbulent flow. This three-dimensional turbulence is shown to produce strong mixing, which is not evident in the rapidly rotating convection regime.

I apply these results to a broad array of planetary dynamo models. Despite the added complexities of a spherical geometry and magnetic fields, the results from plane layer convection hold: I show that heat transfer in the dynamo models exists in two distinct regimes bounded by the transitional Rayleigh number, $Ra_t = E^{-7/4}$. In the rapidly rotating regime, heat transfer is well described by the law $Nu = \left(\frac{Ra}{Ra_c}\right)^{6/5}$, where $Ra_c = 0.046E^{-4/3}$. In the weakly rotating regime, heat transfer follows a $Nu = 0.075Ra^{2/7}$ scaling law. Again, I show that the transition between these regimes is governed by the relative thicknesses of the thermal and velocity boundary layers.

Dynamo models in each of the two heat transfer regimes produce different thermal profiles. The rapidly rotating regime is distinguished by strong, large-scale radial thermal gradients. Models residing in the weakly rotating regime are well mixed, producing nearly isothermal interiors. Thermal mixing is shown to depend on the
boundary layer controlled transition parameter, $Ra/Ra_t$, and is not observed to depend on the Reynolds number. This $Re$-independent mixing may account for the previously unexplained seismic observations of large scale density contrasts in the core. Furthermore, the boundary layer control hypothesis provides a possible explanation for the transition from stable dipolar magnetic field generation to reversing, non-dipolar dynamos. In Earth’s core, which is estimated to reside below, but close to the transition, boundary layer dynamics may be of fundamental importance.
Appendices
A Numerical Methods
A.1 One-Dimensional Thermal Model

Here, I describe the one-dimensional thermal model used to explain the thermal signatures found in rapidly rotating convection experiments in figure 44, section 3.4. The model attempts to simulate a very basic version of heat flow through the physical experiment, with convection treated as idealized heat transport via Taylor columns.

First, a one dimensional model is constructed with dimensions to match the experiment. The model consists of three layers (from bottom to top): a bottom thermal block, which has thickness and thermal conductivity $h_{\text{bottom}}$ and $k_{\text{bottom}}$; a fluid layer with $h_{\text{fluid}}$ and $k_{\text{fluid}}$; and a top thermal block with $h_{\text{top}}$ and $k_{\text{top}}$. As in the experiments, boundary conditions are a fixed heat flux below the bottom thermal block, $q$, and fixed temperature above the top thermal block, $T_{\text{cool}}$.

The basic idea of the model is to vary the efficiency of heat transfer in the fluid with time and solve for the temperature profile at each time step, $t_i$. I assume the convective heat transfer within the fluid is uniform in height, $z$, to approximate the behavior of a Taylor column. This allows me to treat the fluid as a uniform material with effective thermal conductivity $k_{\text{convective}} = k_{\text{fluid}}\nu(t_i)$, where $\nu$ is the Nusselt number. I use a time varying $\nu$ to simulate a location in a three dimensional fluid layer through which Taylor columns pass as they ‘dance’ horizontally around the container. The time series for $\nu$ is taken either from real experimental $\nu$ data, or generated artificially as a stationary time series about a mean value, $\langle \nu \rangle$, taken from the experiments.
The temporal resolution of the model is set to match that of the acquisition frequency of experimental thermal measurements. I assume that the thermal diffusion timescale through the solid thermal blocks is small compared to the typical time scale of fluctuations in convective heat transport. This allows me to solve for a thermally equilibrated temperature profile $T(z, t_i)$ at each time step.

The heat flux and effective thermal conductivity are uniform in space within each layer, so the temperature will vary linearly in $z$ within each layer, and only the temperatures at the boundaries between layers need to solved for. The temperature above the top thermal block is fixed at $T_{cool}$. The temperature between the top thermal block and fluid is $T_3(t_i)$. The temperature between the fluid and the bottom thermal block is $T_2(t_i)$. The temperature at the bottom of the bottom thermal block is $T_1(t_i)$. The aim of the model is to solve for these three temperatures at each time step $t_i$ which depend on the predefined $Nu(t_i)$.

Initially, I take

$$T_3(t = 0) = T_{cool} + \frac{q h_{top}}{k_{top}}$$

(114)

$$T_2(t = 0) = T_3 + \frac{q h_{fluid}}{Nu(t = 0) k_{fluid}}$$

(115)

$$T_1(t = 0) = T_2 + \frac{q h_{bottom}}{k_{bottom}}.$$  

(116)
For the remainder of the time steps, $t_1, \ldots, t_{\text{max}}$, the temperatures are solved for as

\begin{align}
T_3(t_i) &= T_{\text{cool}} + \frac{Nu(t_i) q h_{\text{top}}}{\langle Nu \rangle k_{\text{top}}} \quad (117) \\
T_2(t_i) &= T_3 + \frac{q h_{\text{fluid}}}{Nu(t_i) k_{\text{fluid}}} \quad (118) \\
T_1(t_i) &= T_2 + \frac{q h_{\text{bottom}}}{k_{\text{bottom}}} \quad (119)
\end{align}

Notice that the equation for $T_3(t_i)$ is dependent on fluctuations of $Nu$ about its mean $\langle Nu \rangle$. This is due to the convecting fluid’s control of how much of the base heating $q$ reaches the top thermal block, as the fluid acts as a fluctuating capacitor of heat flux.

Once the temperature profile time series are generated, it is possible to extract a synthetic thermistor time series measurement. In the experiment, the thermistors are located within the thermal blocks, 1 mm from the fluid boundary. Top and bottom thermistor data can then be solved for as

\begin{align}
T_{\text{top thermistor}}(t_i) &= T_3(t_i) - (T_3(t_i) - T_{\text{cool}}) \frac{0.001 \text{ m}}{h_{\text{top}}} \quad (120) \\
T_{\text{bottom thermistor}}(t_i) &= T_2(t_i) + (T_1(t_i) - T_2(t_i)) \frac{0.001 \text{ m}}{h_{\text{bottom}}} \quad (121)
\end{align}

Once the synthetic thermistor data are generated, they can be compared directly to those from experiments. In particular, cross-covariance calculations are made in exactly the same way for the synthetic data as that measured in actual experiments.

The results shown in figure 44 are produced by a model with $h_{\text{bottom}} = 0.015$ m, $k_{\text{bottom}} = 390$ W/mK, $h_{\text{fluid}} = 0.05$ m, $k_{\text{fluid}} = 0.6$ W/mK, $h_{\text{top}} = 0.06$ m, $k_{\text{top}} = 167$ W/mK, $q = 955$ W/m$^2$, $T_{\text{cool}} = 15$ C, and $t_{\text{max}} = 3000$ s, and which use an exper-
imental $Nu$ time series with mean $\langle Nu \rangle = 8.6$ in order to match the experimental setting from which the data in figure 42 are collected.

### A.2 Plane Layer Rotating Convection Simulations

Plane layer numerical simulations were developed by Stephan Stellmach, and I outline his methodology here. Direct numerical simulations are carried out by solving the Boussinesq momentum equation, continuity equation, and energy equation in a cartesian box:

\[
\frac{E}{Pr} \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \mathbf{z} \times \mathbf{u} = -\nabla P + RaE \mathbf{T} \mathbf{z} + E \nabla^2 \mathbf{u} \quad (122)
\]

\[
\nabla \cdot \mathbf{u} = 0 \quad (123)
\]

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T \quad (124)
\]

where $\mathbf{u}$, $P$, and $T$ are the dimensionless velocity, modified pressure, and temperature, respectively, and $\mathbf{z}$ is the vertical unit vector. The fundamental length scale, $L$, is taken to be the height of the fluid layer. The fundamental time scale is taken to be a thermal diffusion time scale, $L/\kappa^2$, where $\kappa$ is the fluid’s thermal diffusivity. The temperature field is normalized by the imposed temperature drop across the layer, $\Delta T$. Three independent, self-similar control parameters arise from this scaling: $Ra$; $E$; and the Prandtl number, $Pr$.

The cartesian box has periodic sidewalls; rigid, no-slip top and bottom boundaries; and diameter to height aspect ratio, $1 \leq \Gamma \leq 4$. Temporal discretization is
accomplished through a second-order, semi-implicit Adams Bashforth backward differ-
entiation time stepping. For spatial discretization, Fourier transform methods are
used in the horizontal direction, while Chebyshev polynomials are used in the vertical
direction in order to resolve better the thin fluid boundary layers.

The parameter range accessed numerically is:

\[ 2 \times 10^3 \leq Ra \leq 10^9 \]  \hspace{1cm} (125)

\[ 10^{-6} \leq E \leq \infty \]  \hspace{1cm} (126)

\[ 1 \leq Pr \leq 100 \]  \hspace{1cm} (127)

Further details on the numerical methods employed are given by Stellmach and

A.3 Dynamo Simulations

The subset dynamo model simulations were carried out by Krista Soderlund, using
a code developed by J. Wicht (Wicht (2002)). I outline the numerical methodology
here.

The model consists of three-dimensional convection of a Boussinesq, electrically-
conducting fluid in a spherical shell rotating about the axial \( \hat{z} \)-direction with fixed
angular velocity \( \Omega \). The shell has isothermal boundaries with an imposed temperature
contrast of \( \Delta T \) between inner and outer boundaries. Gravity is assumed to vary
linearly with radius and has a value of \( g_o \) on the outer boundary. The governing
equations for this system are:

\[
E \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla^2 \mathbf{u} \right) + \mathbf{z} \times \mathbf{u} + \frac{1}{2} \nabla p = \frac{RaE}{Pr} \frac{g(r)}{g_o} T \hat{r} + \frac{1}{2Pm} (\nabla \times \mathbf{B}) \times \mathbf{B},
\]  

(128)

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{Pm} \nabla^2 \mathbf{B},
\]  

(129)

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T,
\]  

(130)

\[
\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0,
\]  

(131)

where \( \mathbf{u} \) is the velocity vector, \( \mathbf{B} \) is the magnetic induction, \( T \) is the temperature, and \( p \) is the non-hydrostatic pressure (Olson et al. (1999), Wicht (2002)). The equations have been non-dimensionalized by shell thickness \( L = R_o - R_i \) as length scale, \( \Delta T \) as temperature scale, \( \tau_\nu = L^2 \nu^{-1} \) as time scale, \( \nu L^{-1} \) as velocity scale, \( \rho \nu \Omega \) as pressure scale, and \( \sqrt{\mu \eta \Omega} \) as magnetic induction scale. In these definitions, \( R_o \) (\( R_i \)) is the outer (inner) shell radius, \( \nu \) is kinematic viscosity, \( \kappa \) is thermal diffusivity, and \( \eta \) is magnetic diffusivity. As dictated by the Boussinesq approximation, density variations are only considered in the buoyancy term (first term on the right hand side of equation 128), and the thermal expansion coefficient, \( \alpha_T \), the kinematic viscosity, the thermal diffusivity, and the magnetic diffusivity are constant (Spiegel (1971)). As mentioned above, the non-dimensional control parameters \( Ra, E, Pr, \) and \( Pm \) are defined in table 1.

Equations (1) - (4) are solved simultaneously for \( \mathbf{u}, \mathbf{B}, \) and \( T \) using the numerical model MagIC version 3.38. This code makes use of the pseudo-spectral method developed by G. Glatzmaier (Glatzmaier (1984)) and subsequently modified by U.
Christensen and J. Wicht (Christensen et al. (1999), Wicht (2002)). The velocity and magnetic induction vectors are decomposed into poloidal and toroidal scalar potentials, which are expanded in Chebyshev polynomials in the radial direction and in spherical harmonic functions on spherical surfaces. MagIC utilizes mixed implicit and explicit time stepping. The Coriolis and nonlinear terms are treated explicitly using a second-order Adams-Bashforth scheme and the diffusion, pressure, and linear terms are treated implicitly using a Crank-Nicolson time step. The implicit time step can vary over time and is limited by a modified MHD CFL criterion which accounts for viscous and ohmic damping of short-wavelength Alfvén-type oscillations (Christensen et al. (1999)). This damping helps to stabilize the system and allows the use of a slightly larger numerical time step compared to the unmodified MHD CFL criterion.

Hyperdiffusion is used in our most extreme cases to increase numerical stability by damping the small-scale components of the flow and magnetic fields. In these cases, the viscous, thermal, and magnetic diffusivities are multiplied by a factor of the form

\[ d(l) = 1 + D \left( \frac{(l + 1 - l_{HD})}{l_{max} + 1 - l_{HD}} \right)^\beta \]  

(132)

where \( d \) is the diffusivity, \( D \) is the hyperdiffusion amplitude, \( l \) is the spherical harmonic degree, \( l_{HD} \) is the degree above which hyperdiffusion starts to act, \( l_{max} \) is the maximum harmonic degree, and \( \beta \) is the hyperdiffusion exponent.
B Table of Experimental Data
Here, I provide a table of the experimental heat transfer data. The column headings have the following meaning: $h$ is the height of the convection tank in meters; RPM is the rotation rate of the tank in rotations per minute; Power is the electrical power supplied to the heater in watts; $Pr$ is the Prandtl number of the fluid; $E$ is the Ekman number; $Ra$ is the Rayleigh number; and $Nu$ is the Nusselt number. Non-dimensional parameters are defined as $Pr = \nu/\kappa$, $E = \nu/2\Omega L^2$, $Ra = \alpha_T g \Delta T L^3/\nu \kappa$, and $Nu = q_{sa} L/k \Delta T$, where $\nu$ is the fluid’s kinematic viscosity, $\kappa$ is the fluid’s thermal diffusivity, $\Omega$ is angular rotation rate, $L (=h)$ is the length scale of the container, $\alpha_T$ is the fluid’s thermal expansivity, $g$ is gravitational acceleration, $\Delta T$ is the temperature drop across the convection tank, $q_{sa}$ is heat flux, and $k$ is the fluid’s thermal conductivity. Water is the working fluid for all cases but those with $h = 9.8 \text{ cm}$, in which sucrose solution was used.

<table>
<thead>
<tr>
<th>$h$</th>
<th>RPM</th>
<th>Power</th>
<th>$Pr$</th>
<th>$E$</th>
<th>$Ra$</th>
<th>$Nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.032</td>
<td>0</td>
<td>10.82</td>
<td>6.946</td>
<td>$\infty$</td>
<td>$1.071 \times 10^6$</td>
<td>8.189</td>
</tr>
<tr>
<td>0.032</td>
<td>0</td>
<td>30.48</td>
<td>6.957</td>
<td>$\infty$</td>
<td>$2.352 \times 10^6$</td>
<td>10.47</td>
</tr>
<tr>
<td>0.032</td>
<td>0</td>
<td>50.59</td>
<td>6.989</td>
<td>$\infty$</td>
<td>$3.479 \times 10^6$</td>
<td>11.62</td>
</tr>
<tr>
<td>0.032</td>
<td>0</td>
<td>100.4</td>
<td>7.051</td>
<td>$\infty$</td>
<td>$5.835 \times 10^6$</td>
<td>13.44</td>
</tr>
<tr>
<td>0.032</td>
<td>0</td>
<td>149.6</td>
<td>6.951</td>
<td>$\infty$</td>
<td>$8.237 \times 10^6$</td>
<td>14.7</td>
</tr>
<tr>
<td>0.032</td>
<td>0</td>
<td>199.9</td>
<td>6.459</td>
<td>$\infty$</td>
<td>$1.183 \times 10^7$</td>
<td>16.18</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>h</th>
<th>RPM</th>
<th>Power</th>
<th>Pr</th>
<th>E</th>
<th>Ra</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.032</td>
<td>0</td>
<td>299.5</td>
<td>5.786</td>
<td>∞</td>
<td>1.953 × 10^7</td>
<td>18.44</td>
</tr>
<tr>
<td>0.032</td>
<td>4.66</td>
<td>10.85</td>
<td>6.933</td>
<td>9.941 × 10^{-4}</td>
<td>9.545 × 10^5</td>
<td>9.251</td>
</tr>
<tr>
<td>0.032</td>
<td>4.66</td>
<td>50.65</td>
<td>7.098</td>
<td>1.015 × 10^{-3}</td>
<td>3.026 × 10^6</td>
<td>12.86</td>
</tr>
<tr>
<td>0.032</td>
<td>4.66</td>
<td>100.6</td>
<td>6.416</td>
<td>9.284 × 10^{-4}</td>
<td>6.259 × 10^6</td>
<td>15.61</td>
</tr>
<tr>
<td>0.032</td>
<td>4.66</td>
<td>200.2</td>
<td>6.812</td>
<td>9.787 × 10^{-4}</td>
<td>9.836 × 10^6</td>
<td>17.28</td>
</tr>
<tr>
<td>0.032</td>
<td>46.63</td>
<td>10.82</td>
<td>6.767</td>
<td>9.724 × 10^{-5}</td>
<td>2.703 × 10^6</td>
<td>3.451</td>
</tr>
<tr>
<td>0.032</td>
<td>46.63</td>
<td>30.48</td>
<td>6.851</td>
<td>9.83 × 10^{-5}</td>
<td>4.083 × 10^6</td>
<td>6.253</td>
</tr>
<tr>
<td>0.032</td>
<td>46.63</td>
<td>50.6</td>
<td>6.738</td>
<td>9.687 × 10^{-5}</td>
<td>5.275 × 10^6</td>
<td>8.351</td>
</tr>
<tr>
<td>0.032</td>
<td>46.63</td>
<td>100.6</td>
<td>7.081</td>
<td>1.012 × 10^{-4}</td>
<td>6.842 × 10^6</td>
<td>11.37</td>
</tr>
<tr>
<td>0.032</td>
<td>46.63</td>
<td>200.3</td>
<td>6.654</td>
<td>9.581 × 10^{-5}</td>
<td>1.126 × 10^7</td>
<td>15.94</td>
</tr>
<tr>
<td>0.032</td>
<td>46.63</td>
<td>299.9</td>
<td>5.916</td>
<td>8.636 × 10^{-5}</td>
<td>1.757 × 10^7</td>
<td>19.64</td>
</tr>
<tr>
<td>0.0472</td>
<td>0</td>
<td>10.81</td>
<td>6.75</td>
<td>∞</td>
<td>3.843 × 10^6</td>
<td>11.54</td>
</tr>
<tr>
<td>0.0472</td>
<td>0</td>
<td>30.47</td>
<td>6.736</td>
<td>∞</td>
<td>8.567 × 10^6</td>
<td>14.67</td>
</tr>
<tr>
<td>0.0472</td>
<td>0</td>
<td>50.59</td>
<td>6.84</td>
<td>∞</td>
<td>1.251 × 10^7</td>
<td>16.09</td>
</tr>
<tr>
<td>0.0472</td>
<td>0</td>
<td>75.06</td>
<td>6.829</td>
<td>∞</td>
<td>1.704 × 10^7</td>
<td>17.6</td>
</tr>
<tr>
<td>0.0472</td>
<td>0</td>
<td>100.3</td>
<td>6.808</td>
<td>∞</td>
<td>2.14 × 10^7</td>
<td>18.85</td>
</tr>
<tr>
<td>0.0472</td>
<td>0</td>
<td>200.4</td>
<td>6.349</td>
<td>∞</td>
<td>4.183 × 10^7</td>
<td>22.54</td>
</tr>
<tr>
<td>0.0472</td>
<td>0</td>
<td>299.2</td>
<td>5.446</td>
<td>∞</td>
<td>7.318 × 10^7</td>
<td>26.12</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>h</th>
<th>RPM</th>
<th>Power</th>
<th>Pr</th>
<th>E</th>
<th>Ra</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0472</td>
<td>0</td>
<td>398.9</td>
<td>4.824</td>
<td>∞</td>
<td>1.123×10^8</td>
<td>28.2</td>
</tr>
<tr>
<td>0.0472</td>
<td>1.91</td>
<td>10.85</td>
<td>6.738</td>
<td>1.087×10^{-3}</td>
<td>3.439×10^6</td>
<td>13.01</td>
</tr>
<tr>
<td>0.0472</td>
<td>1.91</td>
<td>50.35</td>
<td>6.789</td>
<td>1.094×10^{-3}</td>
<td>1.157×10^7</td>
<td>17.63</td>
</tr>
<tr>
<td>0.0472</td>
<td>1.91</td>
<td>101.1</td>
<td>6.907</td>
<td>1.111×10^{-3}</td>
<td>1.969×10^7</td>
<td>19.96</td>
</tr>
<tr>
<td>0.0472</td>
<td>1.91</td>
<td>300.6</td>
<td>5.68</td>
<td>9.348×10^{-4}</td>
<td>6.595×10^7</td>
<td>26.88</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>10.49</td>
<td>6.841</td>
<td>1.102×10^{-4}</td>
<td>4.22×10^6</td>
<td>9.895</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>30.8</td>
<td>6.888</td>
<td>1.108×10^{-4}</td>
<td>7.597×10^6</td>
<td>15.87</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>50.27</td>
<td>6.967</td>
<td>1.12×10^{-4}</td>
<td>1.029×10^7</td>
<td>18.6</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>75.21</td>
<td>6.973</td>
<td>1.12×10^{-4}</td>
<td>1.372×10^7</td>
<td>20.84</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>101</td>
<td>6.98</td>
<td>1.121×10^{-4}</td>
<td>1.709×10^7</td>
<td>22.42</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>200.2</td>
<td>6.823</td>
<td>1.099×10^{-4}</td>
<td>3.043×10^7</td>
<td>26.34</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>300.6</td>
<td>5.873</td>
<td>9.628×10^{-5}</td>
<td>5.46×10^7</td>
<td>30.42</td>
</tr>
<tr>
<td>0.0472</td>
<td>19.1</td>
<td>400.6</td>
<td>5.173</td>
<td>8.607×10^{-5}</td>
<td>8.416×10^7</td>
<td>33.42</td>
</tr>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>10.5</td>
<td>6.806</td>
<td>4.392×10^{-5}</td>
<td>9.41×10^6</td>
<td>4.493</td>
</tr>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>30.53</td>
<td>6.782</td>
<td>4.378×10^{-5}</td>
<td>1.443×10^7</td>
<td>8.59</td>
</tr>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>50.7</td>
<td>6.833</td>
<td>4.407×10^{-5}</td>
<td>1.786×10^7</td>
<td>11.32</td>
</tr>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>75.76</td>
<td>6.84</td>
<td>4.411×10^{-5}</td>
<td>2.159×10^7</td>
<td>13.97</td>
</tr>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>101.1</td>
<td>6.97</td>
<td>4.485×10^{-5}</td>
<td>2.42×10^7</td>
<td>15.9</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>h</th>
<th>RPM</th>
<th>Power</th>
<th>Pr</th>
<th>E</th>
<th>Ra</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>200.3</td>
<td>6.462</td>
<td>4.195×10^{-5}</td>
<td>3.999×10^7</td>
<td>22.67</td>
</tr>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>301.4</td>
<td>5.72</td>
<td>3.767×10^{-5}</td>
<td>6.255×10^7</td>
<td>28.03</td>
</tr>
<tr>
<td>0.0472</td>
<td>47.7</td>
<td>400.5</td>
<td>5.122</td>
<td>3.417×10^{-5}</td>
<td>8.868×10^7</td>
<td>32.28</td>
</tr>
<tr>
<td>0.098</td>
<td>0</td>
<td>20.69</td>
<td>10.59</td>
<td>∞</td>
<td>4.329×10^7</td>
<td>26.52</td>
</tr>
<tr>
<td>0.098</td>
<td>0</td>
<td>50.65</td>
<td>10.03</td>
<td>∞</td>
<td>9.851×10^7</td>
<td>32.45</td>
</tr>
<tr>
<td>0.098</td>
<td>0</td>
<td>100.4</td>
<td>9.623</td>
<td>∞</td>
<td>1.82×10^8</td>
<td>38.19</td>
</tr>
<tr>
<td>0.098</td>
<td>0</td>
<td>200.1</td>
<td>8.803</td>
<td>∞</td>
<td>3.599×10^8</td>
<td>46.29</td>
</tr>
<tr>
<td>0.098</td>
<td>0</td>
<td>299.6</td>
<td>7.518</td>
<td>∞</td>
<td>6.298×10^8</td>
<td>52.92</td>
</tr>
<tr>
<td>0.098</td>
<td>0</td>
<td>496.4</td>
<td>5.986</td>
<td>∞</td>
<td>1.358×10^9</td>
<td>58.32</td>
</tr>
<tr>
<td>0.098</td>
<td>0.5</td>
<td>20.69</td>
<td>10.92</td>
<td>1.512×10^{-3}</td>
<td>3.982×10^7</td>
<td>26.69</td>
</tr>
<tr>
<td>0.098</td>
<td>0.5</td>
<td>50.65</td>
<td>10.86</td>
<td>1.505×10^{-3}</td>
<td>8.197×10^7</td>
<td>32.14</td>
</tr>
<tr>
<td>0.098</td>
<td>0.5</td>
<td>101</td>
<td>10.6</td>
<td>1.472×10^{-3}</td>
<td>1.517×10^8</td>
<td>36.81</td>
</tr>
<tr>
<td>0.098</td>
<td>0.5</td>
<td>249.9</td>
<td>8.578</td>
<td>1.211×10^{-3}</td>
<td>4.543×10^8</td>
<td>48.16</td>
</tr>
<tr>
<td>0.098</td>
<td>5</td>
<td>20.46</td>
<td>10.68</td>
<td>1.483×10^{-4}</td>
<td>3.735×10^7</td>
<td>29.72</td>
</tr>
<tr>
<td>0.098</td>
<td>5</td>
<td>51.04</td>
<td>10.93</td>
<td>1.514×10^{-4}</td>
<td>7.547×10^7</td>
<td>34.61</td>
</tr>
<tr>
<td>0.098</td>
<td>5</td>
<td>101.5</td>
<td>10.75</td>
<td>1.491×10^{-4}</td>
<td>1.361×10^8</td>
<td>39.87</td>
</tr>
<tr>
<td>0.098</td>
<td>5</td>
<td>201.7</td>
<td>9.353</td>
<td>1.312×10^{-4}</td>
<td>3.212×10^8</td>
<td>46.2</td>
</tr>
<tr>
<td>0.098</td>
<td>5</td>
<td>497.2</td>
<td>6.158</td>
<td>8.953×10^{-5}</td>
<td>1.22×10^9</td>
<td>62.37</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>h</th>
<th>RPM</th>
<th>Power</th>
<th>Pr</th>
<th>E</th>
<th>Ra</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.098</td>
<td>25</td>
<td>21.02</td>
<td>10.62</td>
<td>2.949×10⁻⁵</td>
<td>4.121×10⁷</td>
<td>28.07</td>
</tr>
<tr>
<td>0.098</td>
<td>25</td>
<td>50.84</td>
<td>11.01</td>
<td>3.049×10⁻⁵</td>
<td>6.864×10⁷</td>
<td>37.17</td>
</tr>
<tr>
<td>0.098</td>
<td>25</td>
<td>100.8</td>
<td>10.69</td>
<td>2.966×10⁻⁵</td>
<td>1.218×10⁸</td>
<td>44.87</td>
</tr>
<tr>
<td>0.098</td>
<td>25</td>
<td>199.7</td>
<td>9.808</td>
<td>2.74×10⁻⁵</td>
<td>2.472×10⁸</td>
<td>53.64</td>
</tr>
<tr>
<td>0.098</td>
<td>25</td>
<td>497.9</td>
<td>6.458</td>
<td>1.87×10⁻⁵</td>
<td>1.008×10⁹</td>
<td>70.34</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>20.87</td>
<td>10.6</td>
<td>1.472×10⁻⁵</td>
<td>6.278×10⁷</td>
<td>18.38</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>31.03</td>
<td>10.73</td>
<td>1.488×10⁻⁵</td>
<td>7.574×10⁷</td>
<td>22</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>50.91</td>
<td>10.64</td>
<td>1.477×10⁻⁵</td>
<td>1.006×10⁸</td>
<td>27.75</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>75.95</td>
<td>10.86</td>
<td>1.505×10⁻⁵</td>
<td>1.214×10⁸</td>
<td>32.56</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>100.8</td>
<td>10.87</td>
<td>1.506×10⁻⁵</td>
<td>1.425×10⁸</td>
<td>36.78</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>150.8</td>
<td>10.45</td>
<td>1.452×10⁻⁵</td>
<td>1.97×10⁸</td>
<td>43.86</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>201.2</td>
<td>9.655</td>
<td>1.351×10⁻⁵</td>
<td>2.751×10⁸</td>
<td>50.27</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>251.2</td>
<td>8.971</td>
<td>1.262×10⁻⁵</td>
<td>3.625×10⁸</td>
<td>55.57</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>301</td>
<td>8.403</td>
<td>1.189×10⁻⁵</td>
<td>4.528×10⁸</td>
<td>60.53</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>400</td>
<td>7.382</td>
<td>1.056×10⁻⁵</td>
<td>6.82×10⁸</td>
<td>67.29</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>499.5</td>
<td>6.55</td>
<td>9.469×10⁻⁶</td>
<td>9.522×10⁸</td>
<td>73.04</td>
</tr>
<tr>
<td>0.098</td>
<td>50</td>
<td>599</td>
<td>5.855</td>
<td>8.552×10⁻⁶</td>
<td>1.281×10⁹</td>
<td>77.12</td>
</tr>
<tr>
<td>0.197</td>
<td>0</td>
<td>10.82</td>
<td>6.852</td>
<td>∞</td>
<td>2.989×10⁸</td>
<td>43.54</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>h</th>
<th>RPM</th>
<th>Power</th>
<th>Pr</th>
<th>E</th>
<th>Ra</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.197</td>
<td>0</td>
<td>30.73</td>
<td>6.723</td>
<td>∞</td>
<td>6.819×10^8</td>
<td>56.64</td>
</tr>
<tr>
<td>0.197</td>
<td>0</td>
<td>50.58</td>
<td>6.889</td>
<td>∞</td>
<td>9.764×10^8</td>
<td>61.51</td>
</tr>
<tr>
<td>0.197</td>
<td>0</td>
<td>100.8</td>
<td>6.816</td>
<td>∞</td>
<td>1.688×10^9</td>
<td>72.73</td>
</tr>
<tr>
<td>0.197</td>
<td>0</td>
<td>199.8</td>
<td>6.2</td>
<td>∞</td>
<td>3.371×10^9</td>
<td>89</td>
</tr>
<tr>
<td>0.197</td>
<td>0</td>
<td>299.3</td>
<td>5.474</td>
<td>∞</td>
<td>5.578×10^9</td>
<td>103</td>
</tr>
<tr>
<td>0.197</td>
<td>0</td>
<td>401</td>
<td>4.509</td>
<td>∞</td>
<td>1.099×10^10</td>
<td>98.6</td>
</tr>
<tr>
<td>0.197</td>
<td>0.123</td>
<td>10.84</td>
<td>6.937</td>
<td>9.943×10^{-4}</td>
<td>2.916×10^8</td>
<td>43.41</td>
</tr>
<tr>
<td>0.197</td>
<td>0.123</td>
<td>50.67</td>
<td>6.89</td>
<td>9.883×10^{-4}</td>
<td>9.691×10^8</td>
<td>62.06</td>
</tr>
<tr>
<td>0.197</td>
<td>0.123</td>
<td>101</td>
<td>6.842</td>
<td>9.822×10^{-4}</td>
<td>1.676×10^9</td>
<td>72.74</td>
</tr>
<tr>
<td>0.197</td>
<td>0.123</td>
<td>200.1</td>
<td>6.336</td>
<td>9.177×10^{-4}</td>
<td>3.26×10^9</td>
<td>88</td>
</tr>
<tr>
<td>0.197</td>
<td>1.23</td>
<td>10.86</td>
<td>6.898</td>
<td>9.893×10^{-5}</td>
<td>2.771×10^8</td>
<td>46.38</td>
</tr>
<tr>
<td>0.197</td>
<td>1.23</td>
<td>30.83</td>
<td>6.912</td>
<td>9.911×10^{-5}</td>
<td>6.299×10^8</td>
<td>57.66</td>
</tr>
<tr>
<td>0.197</td>
<td>1.23</td>
<td>50.78</td>
<td>6.909</td>
<td>9.907×10^{-5}</td>
<td>9.386×10^8</td>
<td>63.79</td>
</tr>
<tr>
<td>0.197</td>
<td>1.23</td>
<td>101.2</td>
<td>6.873</td>
<td>9.861×10^{-5}</td>
<td>1.623×10^9</td>
<td>74.44</td>
</tr>
<tr>
<td>0.197</td>
<td>1.23</td>
<td>200.2</td>
<td>6.353</td>
<td>9.2×10^{-5}</td>
<td>3.216×10^9</td>
<td>88.73</td>
</tr>
<tr>
<td>0.197</td>
<td>3.075</td>
<td>10.84</td>
<td>6.964</td>
<td>3.991×10^{-5}</td>
<td>2.624×10^8</td>
<td>47.82</td>
</tr>
<tr>
<td>0.197</td>
<td>3.075</td>
<td>30.84</td>
<td>6.947</td>
<td>3.982×10^{-5}</td>
<td>5.938×10^8</td>
<td>60.45</td>
</tr>
<tr>
<td>0.197</td>
<td>3.075</td>
<td>50.7</td>
<td>6.95</td>
<td>3.984×10^{-5}</td>
<td>8.831×10^8</td>
<td>66.74</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>h</th>
<th>RPM</th>
<th>Power</th>
<th>Pr</th>
<th>E</th>
<th>Ra</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.197</td>
<td>3.075</td>
<td>101.1</td>
<td>6.921</td>
<td>$3.969 \times 10^{-5}$</td>
<td>$1.548 \times 10^9$</td>
<td>76.7</td>
</tr>
<tr>
<td>0.197</td>
<td>3.075</td>
<td>200.2</td>
<td>6.416</td>
<td>$3.712 \times 10^{-5}$</td>
<td>$3.054 \times 10^9$</td>
<td>91.47</td>
</tr>
<tr>
<td>0.197</td>
<td>12.3</td>
<td>10.85</td>
<td>6.935</td>
<td>$9.938 \times 10^{-6}$</td>
<td>$2.762 \times 10^8$</td>
<td>45.9</td>
</tr>
<tr>
<td>0.197</td>
<td>12.3</td>
<td>30.83</td>
<td>6.942</td>
<td>$9.946 \times 10^{-6}$</td>
<td>$5.737 \times 10^8$</td>
<td>62.66</td>
</tr>
<tr>
<td>0.197</td>
<td>12.3</td>
<td>50.69</td>
<td>6.967</td>
<td>$9.978 \times 10^{-6}$</td>
<td>$8.353 \times 10^8$</td>
<td>70.14</td>
</tr>
<tr>
<td>0.197</td>
<td>12.3</td>
<td>100.6</td>
<td>6.976</td>
<td>$9.99 \times 10^{-6}$</td>
<td>$1.442 \times 10^9$</td>
<td>80.37</td>
</tr>
<tr>
<td>0.197</td>
<td>12.3</td>
<td>150.6</td>
<td>6.948</td>
<td>$9.953 \times 10^{-6}$</td>
<td>$2.015 \times 10^9$</td>
<td>86.95</td>
</tr>
<tr>
<td>0.197</td>
<td>12.3</td>
<td>201</td>
<td>6.444</td>
<td>$9.314 \times 10^{-6}$</td>
<td>$2.936 \times 10^9$</td>
<td>94.63</td>
</tr>
<tr>
<td>0.197</td>
<td>12.3</td>
<td>299.9</td>
<td>5.509</td>
<td>$8.109 \times 10^{-6}$</td>
<td>$5.221 \times 10^9$</td>
<td>109</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>5.142</td>
<td>6.797</td>
<td>$2.929 \times 10^{-6}$</td>
<td>$4.626 \times 10^8$</td>
<td>13.62</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>10.33</td>
<td>6.84</td>
<td>$2.945 \times 10^{-6}$</td>
<td>$6.037 \times 10^8$</td>
<td>20.66</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>14.94</td>
<td>6.823</td>
<td>$2.939 \times 10^{-6}$</td>
<td>$7.139 \times 10^8$</td>
<td>25.41</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>19.96</td>
<td>6.819</td>
<td>$2.937 \times 10^{-6}$</td>
<td>$8.006 \times 10^8$</td>
<td>30.32</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>24.7</td>
<td>6.91</td>
<td>$2.972 \times 10^{-6}$</td>
<td>$8.634 \times 10^8$</td>
<td>33.73</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>29.94</td>
<td>6.992</td>
<td>$3.003 \times 10^{-6}$</td>
<td>$9.316 \times 10^8$</td>
<td>36.83</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>40.09</td>
<td>7.012</td>
<td>$3.01 \times 10^{-6}$</td>
<td>$1.06 \times 10^9$</td>
<td>43.04</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>50.04</td>
<td>7.037</td>
<td>$3.02 \times 10^{-6}$</td>
<td>$1.178 \times 10^9$</td>
<td>47.91</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>74.76</td>
<td>7.009</td>
<td>$3.009 \times 10^{-6}$</td>
<td>$1.481 \times 10^9$</td>
<td>57.49</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>h</th>
<th>RPM</th>
<th>Power</th>
<th>Pr</th>
<th>E</th>
<th>Ra</th>
<th>Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>101.2</td>
<td>6.991</td>
<td>$3.002 \times 10^{-6}$</td>
<td>$1.783 \times 10^9$</td>
<td>65.08</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>150.6</td>
<td>6.814</td>
<td>$2.935 \times 10^{-6}$</td>
<td>$2.436 \times 10^9$</td>
<td>75.31</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>201</td>
<td>6.418</td>
<td>$2.784 \times 10^{-6}$</td>
<td>$3.017 \times 10^9$</td>
<td>92.88</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>192.2</td>
<td>6.242</td>
<td>$2.717 \times 10^{-6}$</td>
<td>$3.496 \times 10^9$</td>
<td>81.33</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>299.8</td>
<td>5.451</td>
<td>$2.41 \times 10^{-6}$</td>
<td>$5.515 \times 10^9$</td>
<td>105.2</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>284.9</td>
<td>5.297</td>
<td>$2.35 \times 10^{-6}$</td>
<td>$6.422 \times 10^9$</td>
<td>90.52</td>
</tr>
<tr>
<td>0.197</td>
<td>41.01</td>
<td>400.2</td>
<td>4.539</td>
<td>$2.051 \times 10^{-6}$</td>
<td>$1.063 \times 10^{10}$</td>
<td>100.6</td>
</tr>
</tbody>
</table>
References


Chandrasekhar, S. (1953). The instability of a layer of fluid heated below and


Courtillot, V. and Olson, P. (2007). Mantle plumes link magnetic superchrons to


